Applications of Conformal Mappings to Two-Dimensional Electrostatics

Níckolas Alves*
Department of Mathematical Physics, Institute of Physics, University of São Paulo, São Paulo, SP, Brazil
(Dated: September 24, 2019)

Many of the problems in Electrostatics involve finding the electric potential satisfying Laplace's Equation and some boundary conditions. Sometimes, this task might be extremely challenging. The present paper demonstrates a method using conformal mappings to solve two-dimensional electrostatics problems by solving a simpler problem and mapping it to the original one.

Keywords: two-dimensional electrostatics, conformal mappings, analytic functions

I. INTRODUCTION

In the present paper, we are dealing with two-dimensional problems in Electrostatics. You might think this is not a good idea, since our world is three-dimensional, but allow me to remind you that there are problems in three dimensions that can be reduced to a two-dimensional problem. As an example, consider the case of an infinite cylinder with uniform charge density. Even though it is three-dimensional, the problem is symmetric in the cylinder's symmetry axis and we might regard it as a two-dimensional problem without any loss of generality.

That being said, let us begin by recalling Maxwell's Equations:

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}, \\ \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}. \end{cases} \tag{1}$$

When dealing with Electrostatics, Maxwell's Equations reduce to

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}, \\ \nabla \times \mathbf{E} = \mathbf{0}. \end{cases} \tag{2}$$

We haven't written the equations concerning magnetic fields due to the fact that we are not interested in them when studying Electrostatics.

If it also holds that we are interested in a region on space that is free of charge, then Maxwell's Equations become even more simple:

$$\begin{cases} \nabla \cdot \mathbf{E} = 0, \\ \nabla \times \mathbf{E} = \mathbf{0}. \end{cases} \tag{3}$$

Notice that the electric field doesn't have to be identically zero even in this case, because other regions in space could have a non-vanishing charge density. However, we are interested in finding fields on a region respecting the absence of charges.

Notice as well that, since we are in two dimensions, Eq. (3) can be written as

$$\begin{cases} \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} = 0, \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = 0. \end{cases} \tag{4}$$

As a final remark before we properly start, one should notice that the present theory is extremely similar, not to say identical, to the theory of incompressible potential flows in Fluid Mechanics. Some readers might want to have a look at references [1–3], which explain the same ideas in the context of Fluid Mechanics.

II. ELECTROSTATICS AND ANALYTIC FUNCTIONS

A. Electric Potential

When dealing with Electrostatics, the electric field is always curl-less, *i.e.*, $\nabla \times \mathbf{E} = \mathbf{0}$. Thus, in two dimensions, we have (in cartesian coordinates):

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0. \tag{5}$$

It then follows from the Helmholtz Theorem[4] that, provided that $\rho(\mathbf{r})$ vanishes sufficiently fast as $r \to \infty$ and $\mathbf{E} \to 0$ as $r \to \infty$, $\mathbf{E}$ is completely determined by Eq. (2). Furthermore, it holds that

$$\mathbf{E} = -\nabla V, \tag{6}$$

for an electrostatic potential $V(\mathbf{r})$ given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} \, d\tau' , \tag{7}$$

* Correspondence email address: nickolas@fma.if.usp.br

where $\mathscr{r} = \|\boldsymbol{\mathscr{r}}\| = \|\mathbf{r} - \mathbf{r}'\|$. We shall also write $\hat{\boldsymbol{\mathscr{r}}} \equiv \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}}$ in the following sections.

One might find it weird that we are integrating ρ if we are studying the electric field in the absence of any charges. Nevertheless, even though we want to figure out the field in a chargeless region, it doesn't mean that there are no charges on other regions of space.

The importance of this concept is that if we are able to obtain the expression for $V(\mathbf{r})$, the electric field (which is the physical quantity we are interested in) is one derivative away. This is certainly interesting, because if once we were facing the challenge of calculating a vector field, we now only have to find a scalar field and differentiate in order to obtain the answer we are looking for.

Finally, we note for further use that

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}'. \tag{8}$$

For a proof of this expression, you might see [4, 5].

B. Field Lines

In space free of any charges, the electric field is divergenceless, *i.e.*, $\nabla \cdot \mathbf{E} = 0$. Thus, we have the following equation (in cartesian coordinates):

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0. \tag{9}$$

Suppose there was a function $U(\mathbf{r})$ such that[1, 3]

$$E_x = -\frac{\partial U}{\partial y}, \quad E_y = \frac{\partial U}{\partial x}. \tag{10}$$

Then it follows that

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = -\frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 U}{\partial y \partial x} = 0, \tag{11}$$

and the condition that $\nabla \cdot \mathbf{E} = 0$ is automatically satisfied. Furthermore, if we knew U, we would be a derivative away from $\mathbf{E}$.

For now, we are going to focus on finding out wheter U exists or not. After all, there would be no sense to derive results about a function that does not exist. Therefore, we are facing the problem of solving the following partial differential equation system:

$$\begin{cases} \dfrac{\partial U}{\partial x} = E_y(x, y), \\ \dfrac{\partial U}{\partial y} = -E_x(x, y), \end{cases} \tag{12}$$

where the functions $E_x(x, y)$ and $E_y(x, y)$ are known.

From the first equation in Eq. (12), we have

$$U(x, y) = \int_{x_0}^{x} E_y(x', y)\, dx' + f(y), \tag{13}$$

for some arbitrary function $f(y)$. If we differentiate Eq. (13) with respect to y, we get

$$\frac{\partial U}{\partial y} = \int_{x_0}^{x} \frac{\partial E_y}{\partial y}\, dx' + f'(y). \tag{14}$$

By imposing the condition $\nabla \cdot \mathbf{E} = 0$, it follows that

$$\begin{aligned} \frac{\partial U}{\partial y} &= -\int_{x_0}^{x} \frac{\partial E_x}{\partial x'}\, dx' + f'(y), \\ &= -E_x(x, y) + E_x(x_0, y) + f'(y). \end{aligned} \tag{15}$$

The second equation on Eq. (12) guarantees then that $f'(y) = -E_x(x_0, y)$. Integrating this ODE allows us to conclude that

$$f(y) = -\int_{y_0}^{y} E_x(x_0, y')\, dy'. \tag{16}$$

Substituting this result in Eq. (13) gives us a final expression for U up to the choice of arbitrary constants x_0 and y_0:

$$U(x, y) = \int_{x_0}^{x} E_y(x', y)\, dx' - \int_{y_0}^{y} E_x(x_0, y')\, dy'. \tag{17}$$

Thus, there is indeed such a function U. But why would such a function be interesting?

Notice that, if we regard U as a three-variable function constant with respect to z, then

$$\begin{aligned} E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} &= -\frac{\partial U}{\partial y}\hat{\mathbf{x}} + \frac{\partial U}{\partial x}\hat{\mathbf{y}}, \\ &= \hat{\mathbf{z}} \times \nabla U, \\ \mathbf{E} &= -\nabla U \times \hat{\mathbf{z}}. \end{aligned} \tag{18}$$

Since U is a two-variable function, ∇U is orthogonal to $\hat{\mathbf{z}}$ and lies in the xy-plane. The same goes to $\mathbf{E}$, as we can see from the cross-product in Eq. (18). We see then that everywhere in the xy-plane $\mathbf{E}$ and ∇U are orthogonal vectors. We also know, from multivariable calculus, that ∇U is always orthogonal to level curves of U. However, since we are in two-dimensions, it follows then that $\mathbf{E}$ is always parallel to such curves[1][3].

Moreover, we now see that the constants x_0 and y_0 that appeared in Eq. (17) are related to which field line is described by the curve $U(x, y) = 0$.

---

[1] Ever heard that the enemy of my enemy is my friend? I guess it holds in two dimensions, since the orthogonal to my orthogonal is my parallel. The proverb fails in three dimensions though (*e.g.*, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$).

C. Complex Potential

We can now express **E** in two different manners:

$$\begin{cases} \mathbf{E} = -\boldsymbol{\nabla} V, \\ \mathbf{E} = -\boldsymbol{\nabla} U \times \hat{\mathbf{z}}. \end{cases} \quad (19)$$

Therefore, the following conditions hold for the functions V and U associated to a certain electric field:

$$\begin{cases} \frac{\partial V}{\partial x} = \frac{\partial U}{\partial y}, \\ \frac{\partial V}{\partial y} = -\frac{\partial U}{\partial x}. \end{cases}$$

Id est, the functions V and U satisfy the Cauchy-Riemann Conditions for analytic functions[6].

If we take the derivatives with respect to both variables x and y of both equations in Eq. (20), we can obtain with some algebra that

$$\begin{cases} \nabla^2 V = 0, \\ \nabla^2 U = 0, \end{cases} \quad (20)$$

and therefore both U and V are harmonic functions.

This seems useless, albeit curious. Can we achieve anything new with such information?

In fact, yes, we can. I remind you of the following theorems from complex calculus:

Theorem 1:

Let $W(z) = V(x, y) + iU(x, y)$ be a function for $x + iy = z \in S \subseteq \mathbb{C}$. Suppose $W'(z_0)$ exists for some $z_0 \in S, z_0 = x_0 + iy_0$. Then the first partial derivatives of V and U exist at z_0 and satisfy the Cauchy-Riemann conditions (Eq. (20)). Furthermore, the derivative of W at z_0 is given by

$$W'(z_0) = \frac{\partial V}{\partial x} + i\frac{\partial U}{\partial x}, \quad (21)$$

and, equivalently,

$$W'(z_0) = \frac{\partial U}{\partial y} - i\frac{\partial V}{\partial y}. \quad (22)$$

□

Proof:

The derivative of W at z_0 is, by definition,

$$W'(z_0) = \lim_{z \to z_0} \frac{W(z) - W(z_0)}{z - z_0}. \quad (23)$$

We know[6] that limits in the complex plane can be regarded as limits in $\mathbb{R}^2$ and as independent limits in the real and imaginary parts. Furthermore, for a limit in $\mathbb{R}^2$ to converge, it must have the same value when we reach the limiting point through any curve. Since, by hypothesis, $W'(z_0)$ exists, we might choose any path to calculate the limit. If we pick the path $x = x_0$, while varying y, we get

$$\begin{cases} \mathrm{Re}[W'(z_0)] = \lim_{y \to y_0} \frac{U(x_0, y) - U(x_0, y_0)}{y - y_0}, \\ \qquad = \frac{\partial U}{\partial y}(x_0, y_0), \\ \mathrm{Im}[W'(z_0)] = \lim_{y \to y_0} -\frac{V(x_0, y) - V(x_0, y_0)}{y - y_0}, \\ \qquad = -\frac{\partial V}{\partial y}(x_0, y_0), \end{cases} \quad (24)$$

A similar calculation with $y = y_0$, varying x, yields

$$\begin{cases} \mathrm{Re}[W'(z_0)] = \lim_{x \to x_0} \frac{V(x, y_0) - V(x_0, y_0)}{x - x_0}, \\ \qquad = \frac{\partial V}{\partial x}(x_0, y_0), \\ \mathrm{Im}[W'(z_0)] = \lim_{x \to x_0} \frac{U(x, y_0) - U(x_0, y_0)}{x - x_0}, \\ \qquad = \frac{\partial U}{\partial x}(x_0, y_0), \end{cases} \quad (25)$$

Since $\frac{\partial V}{\partial x}(x_0, y_0) = \mathrm{Re}[W'(z_0)] = \frac{\partial U}{\partial y}(x_0, y_0)$ and $-\frac{\partial V}{\partial y}(x_0, y_0) = \mathrm{Im}[W'(z_0)] = \frac{\partial U}{\partial x}(x_0, y_0)$, the Cauchy-Riemann conditions are indeed satisfied. Furthermore, Eq. (24) shows that

$$W'(z_0) = \frac{\partial U}{\partial y} - i\frac{\partial V}{\partial y},$$

while Eq. (25) gives us

$$W'(z_0) = \frac{\partial V}{\partial x} + i\frac{\partial U}{\partial x},$$

proving the theorem. ■

This shows us that the Cauchy-Riemann conditions are necessary for a complex function to be differentiable at some point. We might as well use them to obtain a sufficient condition for a complex function to be differentiable.

Theorem 2:

Let $W(z) = V(x, y) + iU(x, y)$ be a function for $x + iy = z \in S \subseteq \mathbb{C}$. Let $z_0 \in S$ and let $\mathcal{A} \subseteq S$ be some neighborhood of z_0. Suppose that U and V are both differentiable with respect to both x and y in $\mathcal{A}$, these derivatives being continuous at z_0. Then, if the Cauchy-Riemann conditions are met at z_0, $W'(z_0)$ exists. □

For the proof of this theorem, see [6].

In general, the electric field is continuous in free space, *i.e.*, when charges are absent. Thus, the derivatives of U and V exist and are continuous at every point we are currently interested, and thus $W(z) = V(x,y) + iU(x,y)$ is differentiable at every point we are interested. You might recall from complex calculus that functions that are differentiable on a neighborhood of a point z_0 are said to be *analytic* at z_0. Thus W is an analytic function on the regions we are interested.

Concerning Electrostatics, we see now that it might be interesting to define a complex field[5] $E = E_x - iE_y$ and make good use of the fact that

$$E = -W'(z). \tag{26}$$

However, notice one more thing: we could also have picked $-U$ instead of U, changing only the signs in every expression and leading us to the complex potential $W_1(z) = -U(x,y) + iV(x,y)$, reversing whether U or V is the real (imaginary) component of W. Since both U and V always respect Laplace's Equation, any of them can stand for the electric potential and any of them can stand for the field lines. Thus, obtaining a single complex potential solves two problems at once: the one in which V is the electric potential and the one in which U is the electric potential.

Of course, although we have built the complex potential in cartesian coordinates, we could also have done it in polar coordinates. The expressions would simply be[1]:

$$\begin{cases} E_r = -\dfrac{\partial V}{\partial r} = -\dfrac{1}{r}\dfrac{\partial U}{\partial \theta}, \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial V}{\partial \theta} = \dfrac{\partial U}{\partial r}. \end{cases} \tag{27}$$

III. SOME EXAMPLES

Now that we have some idea about how our theory looks like, we should give some examples; especially considering that this could give us information on some simple potentials that could be combined to solve harder problems. For example, we are able to describe the potential of a charge under an uniform electric field by simply adding both potentials. As we shall see, this can quickly become a powerfull tool.

A. Uniform Electric Field

Suppose we want to describe a uniform electric field $\mathbf{E} = E_x\mathbf{x} + E_y\mathbf{y}$, where E_x and E_y are both constants.

Solving the PDEs given in Eq. (19) (or using Eqs. (7) and (17)) we obtain, up to an arbitrary constant of integration[1],

$$\begin{cases} V(x,y) = -E_x x - E_y y; \\ U(x,y) = E_y x - E_x y, \end{cases} \tag{28}$$

and thus

$$W(z) = -Ez. \tag{29}$$

We could, of course, leave the arbitrary constant of integration explicitly on the expression for the potential. However, it would vanish as soon as we take a derivative in order to get the electric field (which is the quantity we are actually interested). Therefore, the integration constant is superfluous.

The electric potential and the field lines for this configuration are shown in Figure 1.

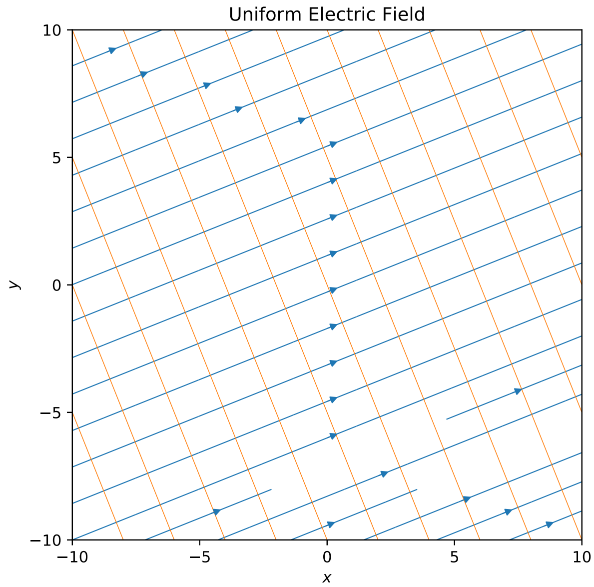

Figure 1. Electric potential and field lines for an uniform electric field $\mathbf{E} = \mathbf{5\hat{x}} + \mathbf{2\hat{y}}$. The orange lines are the level curves for the electric potential, while the blue lines represent the field lines.

B. Quadrupole

Lets now examine another approach: given an analytic function, we want to examine what is the physical problem described by it.

Consider the function[5] $W(z) = -\frac{1}{2}z^2$. Through some algebra, you can find out that

$$\begin{cases} V(x,y) = -\frac{1}{2}\left(x^2 - y^2\right); \\ U(x,y) = -xy. \end{cases} \tag{30}$$

Thus, we see that the equipotentials are hyperbolas, and the same goes to the field lines (though they are different hyperbolas).

Plotting the equipotentials and field lines, we obtain Figure 2. If you pay atention, you will realize this field corresponds to the one generated by a quadrupole configuration. We might then pick some equipotentials and regard them as the boundary of charged conductors responsible for the creation of such a field[5]. After all, the real problem involves the presence of such conductors and they are indeed equipotentials[4].

Nevertheless, there is an important remark to be made about this description. As soon as we pick an equipotential as the boundary of a conductor, the interior of such conductor is no longer described by our potential. Indeed, conductors are equipotentials as a whole, not only on their surfaces, and the potential we found is not constant on the inside of the conductor. It describes the physics observed on the outside of the conductor, where space is free of charge up to its boundary, but the description ends there.

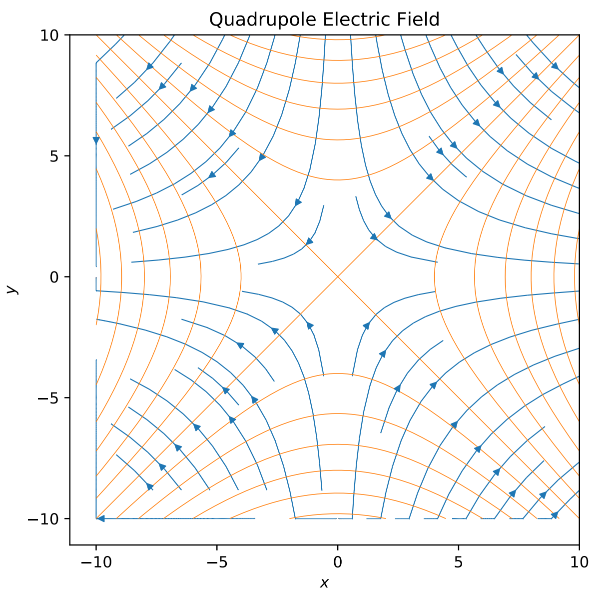

Figure 2. Electric potential and field lines for a quadrupole $\mathbf{E} = x\hat{\mathbf{x}} - y\hat{\mathbf{y}}$. The orange lines are the level curves for the electric potential, while the blue lines represent the field lines.

C. Heaviside Step Function

Let us now consider the function given by $W(z) = -\frac{i}{\pi}\log z$. Using the polar form of complex numbers, $z = re^{i\theta}$, we have that $W(z) = -i\frac{1}{\pi}\log r + \frac{1}{\pi}\theta$. Therefore,

$$\begin{cases} V(r,\theta) = +\frac{1}{\pi}\theta; \\ U(r,\theta) = -\frac{1}{\pi}\log r. \end{cases} \tag{31}$$

Let $x \in \mathbb{R}$. For $x < 0$, we see that $V(x,0) = 1$, while for $x > 0$ we have $V(x,0) = 0$. Thus, this potential solves a boundary value problem similar to Heaviside's step function. Actually, $V(x,0) = H(-x)$, where H stands for Heaviside's step function.

The equipotentials and field lines are plotted in Figure 3.

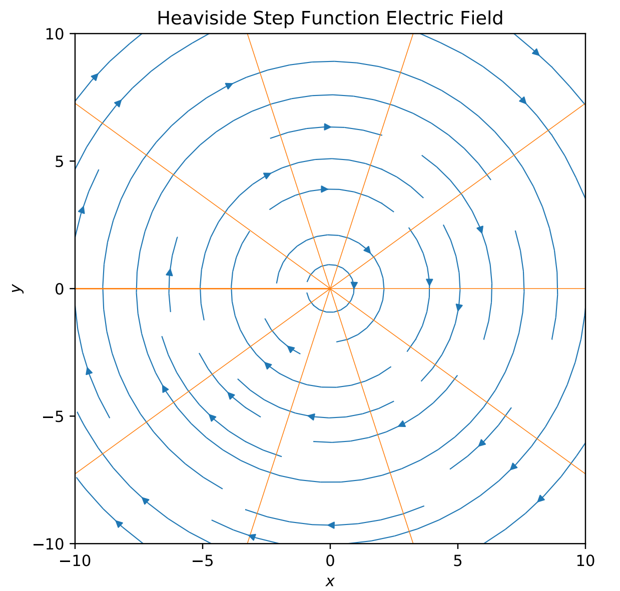

Figure 3. Electric potential and field lines for boundary value problem with $V(x,0) = H(-x)$, where H denotes Heaviside's step function. The orange lines are the level curves for the electric potential, while the blue lines represent the field lines.

To be completely fair, either the logarithm is analytic in all $\mathbb{C}$ or it is a function. Choose one. The reason for this is, essentially, that $z = re^{i\theta} = re^{i(\theta+2\pi)}$, and thus the complex logarithm is not single-valued. We must then choose a branch (*i.e.*, a restrained range) for the logarithm to be analytic. Usually we give up the negative real axis, since we didn't have it in the real case. However, for our problem, we could simply pick a branch that is cut at the negative y values, so we keep analyticity on the upper half-plane. After all, a bird in the hand is worth two in the bush, and in this way our problem is solved for the upper half-plane.

IV. CONFORMAL MAPPINGS

Even though it is really cool to write the potentials and field line functions, it certainly isn't so clear yet what is so great about analytic functions and the complex potential. Ok, we could solve some problems accidentally by differentiating analytic functions at random, but did we gain anything with this new strategy?

What is actually really powerful about analytic functions is that they generate conformal mappings in $\mathbb{C}$. This is fancy language for saying that they preserve angles of intersection when we apply them to the complex plane as a whole[5].

For example, suppose we have some equipotentials and field lines on the complex plane. We already know they always intersect at right angles (this follows from Eq. (19) and the fact that gradients are always orthogonal to level curves). If we map the complex plane into itself (or into another copy of the complex plane, if you'd rather think this way) with a conformal map, the mapped curves would *still* be crossing at right angles. Essentially, they preserve the local *form* of the space. Hence, conformal[1, 5].

Formally, we say that a mapping is conformal at a certain point whenever it preserves both the magnitude and sense of the angle between any two smooth arcs crossing each other at that point[6].

Great, but what does this has to do with analytic functions? Well, as I said before, analytic functions *are* conformal mappings. Or, strictly speakig, the following theorem holds[6]:

Theorem 3:
Let $f : S \subseteq \mathbb{C} \to \mathbb{C}$ *be an analytic function at a point* $z_0 \in S$*. Then, if* $f'(z_0) \neq 0$*, the map* $w = f(z)$ *is conformal.* □

Proof:
Let C be a smooth arc[2] passing through z_0 and parametrically represented by $z(t) = x(t) + iy(t)$, for $a \leqslant t \leqslant b$. Then the image of C through f, lets call it Γ, is parametrically represented by $w(t) = f[z(t)]$, again for $a \leqslant t \leqslant b$. The chain rule guarantees that

$$w'(t) = f'[z(t)] \cdot z'(t). \tag{32}$$

Thus, since $f'(z_0) \neq 0$, Γ is also a smooth arc, at least on a neighborhood of z_0.

If we consider both sides of Eq. (32) in their polar form, we know them both must have the same argument, *i.e.*,

$$\arg w'(t) = \arg f'[z(t)] + \arg z'(t). \tag{33}$$

However, $\arg z'(t)$ is the angle that C makes with the x-axis at the point $z(t)$! This might become a bit clearer if you think in $\mathbb{R}^2$ (and, perhaps, take a look at Figure 4 to help your reasoning): if we have a parametric curve in $\mathbb{R}^2$, its derivative at a point is the vector tangent to the curve at that point. Well, in $\mathbb{C}$ this vector is precisely a complex number, which we may represent in either cartesian coordinates or polar coordinates. In its polar form, the angle that the vector (and thus the curve) makes with the horizontal axis is simply the vector's argument. Hence the statement.

Let $\phi_0 \equiv \arg w'(t_0)$, $\psi_0 \equiv \arg f'[z(t_0)]$ and $\theta_0 \equiv \arg z'(t_0)$, where t_0 is such that $z_0 = z(t_0)$. Of course $\phi_0 = \psi_0 + \theta_0$ (this is just a special case of Eq. (33)). Thus, the smooth arc was simply locally rotated by ψ_0. The argument applies to every smooth arc passing through z_0, *i.e.*, every smooth arc passing through z_0 is rotated by the exact same amount and in the same direction. Thus, the angle at which their images cross at the point $f(z_0)$ is equal, in both magnitude and sense, to the angle at which the original arcs crossed each other at the point z_0. ■

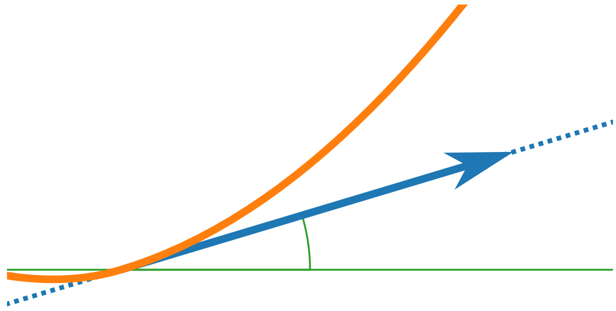

Figure 4. In blue, the tangent $z'(t)$ to a smooth arc $z(t)$ (orange). Notice that the blue vector is just a complex number and the angle it makes with the horizontal (green) is this number's argument.

When I say the *sense* of the angle is preserved, I mean that the two curves can't "change roles". They are simply rotated, not reflected. For example, in Figure 4, suppose that the angle between the green line and the

[2] A set C of points in the complex plane is said to be an *arc* when we can describe it parametrically it as $z(t) = x(t) + iy(t)$, where x and y are continuous functions of the real parameter $t \in [a, b]$. An arc is said to be *smooth* when $z(t)$ is continuously differentiable in $[a, b]$ and $z'(t) \neq 0, \forall\, t \in [a, b]$.

orange curve (or the blue tangent vector, equivalently) is α when we count it counterclockwise starting from the green line. The transformation preserves the angle's magnitude and sense when, after transforming the curves, the angle between both curves is still α when I still count it counterclockwise starting from the green line. If I had to start from the orange line, or count in clockwise, the sense would not have been preserved[3].

Again, this is all great. But why is it useful?

Suppose you are facing a particularly difficult two-dimensional problem. You know you can solve it if you find the complex potential (or simply the electric potential, or even the field line function), but that doesn't mean it is easy to find that potential. However, suppose you can map that difficult problem conformally to a simple one. Now you just gotta solve the simple problem, reverse the mapping and voilà: you have the complex potential for the tricky problem[5]!

So here's our plan: first of all, pick your problem. We'll say it is in the $\zeta = \chi + i\eta$ space, where the potentials are ϕ (electric potential), ψ (field lines) and $\omega(\zeta) = \phi(\chi, \eta) + i\psi(\chi, \eta)$ is the complex potential. Suppose you can use some conformal mapping f to map this space into another space that makes the problem easier to solve. We shall write it $z = f(\zeta)$. We find the complex potential $W(z) = V(x, y) + iU(x, y)$ in this simple space and then pull everything back: $\omega(\zeta) = W(f(\zeta))$.

Perhaps you are wondering: why the hell would this work? In principle, it doesn't need to. However, if we define ω through the expression $\omega(\zeta) = W(f(\zeta))$, it follows that ω is an analytic function and thus it is a complex potential in the ζ space. Thus, it solves some electrostatic problem over there. If we can get the boundary conditions just right, then the uniqueness of solution to Laplace's Equation[4] guarantees we got the right solution.

A. Bending a Plane

Just to be sure we are getting things right, lets start with some problem we can solve through some other method so we know our solution is correct. Suppose we have two infinite uniformly charged planes crossing each other at a right angle. We want to find the electric field at some point outside of both planes.

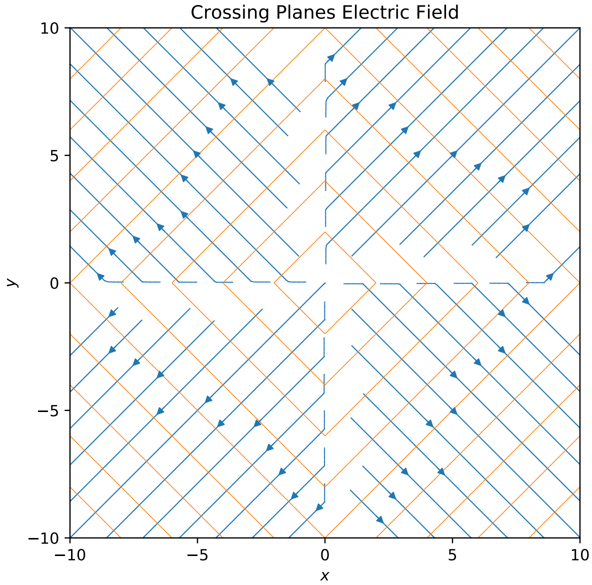

Figure 5. Field lines (blue) and equipotentials (orange) for two orthogonal infinite planes with uniform charge $\sigma = 2\epsilon_0$.

Due to Gauss's Law, we know that the field due to an infinite uniformly charged plane resting in the xz-plane is given by

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma}{2\epsilon_0} \frac{y}{|y|} \hat{\mathbf{y}} \tag{34}$$

where σ stands for the plane's surface charge density.

If we have another identical plane resting at the yz-plane, then the resulting field due to both planes is, due to the Superposition Principle,

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma}{2\epsilon_0} \left(\frac{x}{|x|} \hat{\mathbf{x}} + \frac{y}{|y|} \hat{\mathbf{y}} \right). \tag{35}$$

The electric potential is given by

$$V(x, y) = -\frac{\sigma}{2\epsilon_0} \left(|x| + |y| \right). \tag{36}$$

Both the field lines and equipotentials are shown in Figure 5.

This would be the usual solution for the problem. However, we want to test our conformal plan. Instead of trying to solve the problem on all space, lets pick something slightly easier: just a quarter. It makes sense to try this strategy: the solution we found through Gauss's Law isn't analytic when x or y is zero (and there's charge over there!). However, if we have simply a corner, we can rotate it, reflect it and use the Superposition Principle to recreate the two intersecting planes. This is our "difficult" space, ζ.

---

[3] Some maps might preserve the angle's magnitude, while not preserving it's sense. For an example, the map $w = \bar{z}$ preserves the angle's magnitude, but not its sense. Such maps are called isogonal mappings[6].

Lets then start with an uniform field, to which the potential is given by $W(z) = -E_0 z$ (Eq. (29)), where both z and E_0 are complex numbers. We will pick a plane orthogonal to the y axis, and thus $W(z) = iE_y z$. Now the equipotentials are curves with $V(x, y) = -E_y y$ constant, and thus we might pick the horizontal plane $y = 0$ as the charged plane generating the field. In doing this, we are giving up on the lower half-plane, because it is no longer physical (notice that the field lines over there point towards the charged plane, instead of being repelled by it). This space is the "easy" space, z, in which we already know how to solve the potential problem.

We want to bend this plane into a $\frac{\pi}{2}$ radians corner, which is the hard problem. I claim that the function $z = f(\zeta) = \zeta^2$ (which is analytic!) does the job.

Indeed, let $\zeta = i\eta, \eta \in \mathbb{R}_+$. Then $z = f(\zeta) = (i\eta)^2 = -\eta^2$. Since $\eta \in \mathbb{R}, \eta^2 > 0$ and $-\eta^2 \in \mathbb{R}_-$. Furthermore, if we let $\zeta = \chi \in \mathbb{R}_+$, $f(\zeta) = \chi^2 \in \mathbb{R}_+$. Thus, f maps the difficult corner problem into the easy plane one.

Last step: $\omega(\zeta) = W(f(\zeta)) = W(\zeta^2)$. Therefore, we have

$$\omega(\zeta) = iE_y \zeta^2. \tag{37}$$

For our problem, we might pick $E_y = \frac{\sigma}{4\epsilon_0}$. We are going to superpose more corners afterwards, and thus we should now pick just half of the final charge. Therefore,

$$\omega(\zeta) = i\frac{\sigma}{4\epsilon_0}\zeta^2. \tag{38}$$

The complex electric field is given by $E = -\omega'(\zeta)$.

$$E = \frac{\sigma}{2e}\eta - i\frac{\sigma}{2e}\chi. \tag{39}$$

The solutions for the field lines and electric potential obtained through this method are plotted in Figure 6. As you can see, the solution doesn't seem so be similar to what we found through Gauss's Law.

However, remember that we were not solving the same problem. We used Gauss's Law to solve the problem of crossing infinite planes, but we used conformal maps to solve the problem of a charged corner. Furthermore, we always knew our solution was going to be physical only for $\chi, \eta > 0$, because that's where we mapped the physical solutions for the simple problem (below the charged plane in the simple problem, the potential means nothing for a physicist). Thus, if we want to get the solution to crossing planes, we still have to compute what happens *outside* of the corner.

Curiously, we can use the very same simple problem, just map it in another way. In general, conformal mappings of the form $f(\zeta) = a\zeta^k$ can solve pretty much any corner/wedge problem[1], and that's exactly how we are going to face the problem of finding the electric field outside of the conducting corner.

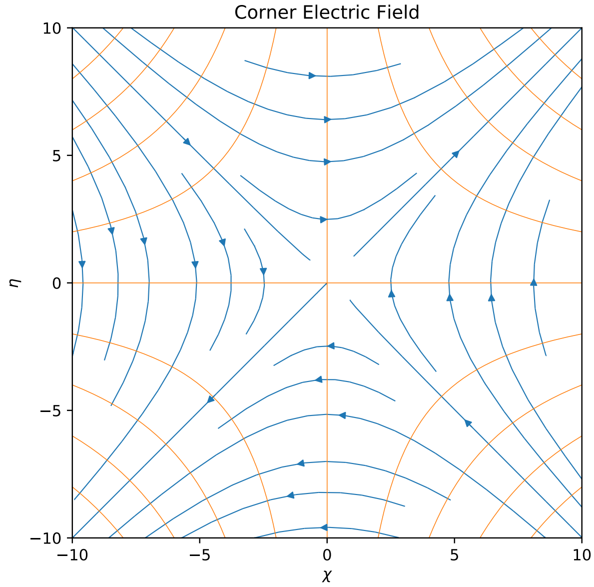

Figure 6. Field lines (blue) and equipotentials (orange) for the bent plane. Although the solution for $\chi, \eta > 0$ seems something like the solutions we got for the crossing planes through Gauss's Law, it is not what we expected. We can't forget that this problem is not the same problem, and the region outside $\chi, \eta > 0$ is non-physical for this solution.

Now we need to find a conformal map that leaves us just the portion of the complex plane that has at least one coordinate with a negative value. If we can map the upper complex plane (which is the physical portion of our simple problem) into any three-quarters of the complex plane, we can then simply rotate what we got and we will be done.

If we simply take powers of a complex number, we are making rotations in the complex plane. Moreover, positive real numbers do not rotate at all (their argument is 0). Since we need to map three-quarters of the complex plane into the upper half, a nice guess would be to leave the fourth quarter outside of the game and try rotating the plane clockwise, until the negative imaginary axis coincides with the negative real axis.

Therefore, we are looking for a function of the form $z = f(\zeta) = \zeta^k$ such that $\zeta = -i\eta, \eta \in \mathbb{R}_+$ gets mapped to some point $-x, x \in \mathbb{R}_+$. Using the polar form of complex numbers, we see that such a transformation must satisfy:

$$z = re^{i\pi} = |\zeta|^k e^{\frac{3ik\pi}{2}} = \zeta^k. \tag{40}$$

Thus, we see that $i\pi = \frac{3ik\pi}{2}$ and it follows that

$k = \frac{2}{3}$, giving us the transformation $f(\zeta) = \zeta^{\frac{2}{3}}$. However, don't forget we had to rotate the plane so the pieces would fit and the non-physical region of this mapping would be on the first quarter (then we can simply stick both solutions together and get the final result). Since currently the non-physical region is in the fourth quarter, we must rotate the complex plane by $\frac{\pi}{2}$ radians counterclockwise and *then* bend the plane. Such a rotation is described by the transformation $z = g(\zeta) = e^{-i\frac{\pi}{2}}\zeta = -i\zeta$. Our final transformation is then just the composition of both these transformations, *i.e.*

$$z = f(\zeta) = (-i\zeta)^{\frac{2}{3}} . \tag{41}$$

Lets check: pick $\zeta = i\eta, \eta \in \mathbb{R}_+$.

$$\begin{aligned} f(\zeta) &= (-i \cdot i\eta)^{\frac{2}{3}} , \\ &= \eta^{\frac{2}{3}} \end{aligned} \tag{42}$$

Pick now $\zeta = \chi \in \mathbb{R}_+$.

$$\begin{aligned} f(\zeta) &= (-i\chi)^{\frac{2}{3}} , \\ &= \left(e^{\frac{3\pi}{2}}\chi\right)^{\frac{2}{3}} , \\ &= -\chi^{\frac{2}{3}} , \end{aligned} \tag{43}$$

This function maps positive real numbers into negative real numbers and positive imaginary numbers into positive real numbers, exactly as we needed. The upper half-plane is mapped to all the complex-plane, except the first quarter (which we already have). Thus, we can now find the complex potential:

$$\begin{aligned} \omega(\zeta) &= W(f(\zeta)), \\ &= i\frac{\sigma}{4\epsilon_0} f(\zeta), \\ &= -i\frac{\sigma}{4\epsilon_0}\zeta^{\frac{2}{3}} . \end{aligned} \tag{44}$$

[1] F. Grassi, "Mecânica dos fluidos para físicos: Uma introdução," (2019), unpublished.

[2] L. D. Landau and L. M. Lifshitz, *Fluid Mechanics*, Course of Theoretical Physics, Vol. 6 (Pergamon Press, Oxford, 1987).

[3] A. R. Paterson, *A First Course in Fluid Dynamics* (Cambridge University Press, Cambridge, 1983).

[4] D. J. Griffiths, *Introduction to Electrodynamics* (Cambridge University Press, Cambridge, 2017).

[5] A. Zangwill, *Modern Electrodynamics* (Cambridge University Press, Cambridge, 2013).

[6] R. Churchill, *Complex Variables and Applications* (McGraw-Hill, New York, 1974).