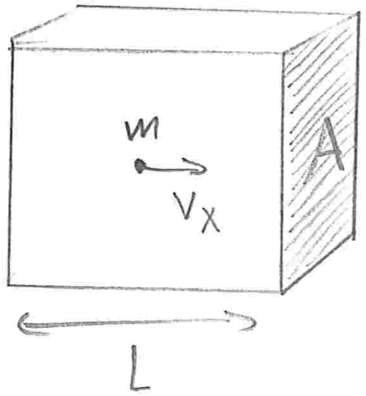
Classical Transport Phenomena

1. Naive Kinetic Theory (Clausius, Maxwell, Boltzmann

The general postulates are the following:

- Very large number of molecules, but large separation compared to size;
- Particles move around randomly with a certain velocity distribution;
- Collisions between particles and with walls are perfectly elastic, and there are no other interactions,
- Particles obey Newton's Laws.

1.1. The particle in a box



$$\Delta p_x = 2mv_x, \quad \Delta t = \frac{2L}{v_x}$$

$$F = \frac{mv_x^2}{L} \Rightarrow p = \frac{mv_x^2}{LA} = \frac{mv_x^2}{V}$$

$$\Rightarrow pV = mv_x^2$$

For $N \gg 1$ particles we expect $pV = nRT = Nk_BT$ and therefore

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \frac{k_BT}{m}.$$

1.2. The Maxwell-Boltzmann distribution

We impose a few constraints on the velocity distribution $F(\vec{v})$.

* Spherical symmetry:

$$F(\vec{v}) = F(v^2)$$

* Independent components:

$$F(\vec{v}) = f_x(v_x) f_y(v_y) f_z(v_z)$$

These considerations together with differentiability fix a gaussian

$$f_x(v_x) = f(v_x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{v_x^2}{2\sigma^2}}.$$

From the particle in a box we conclude $\sigma^2 = \frac{k_B T}{m}$

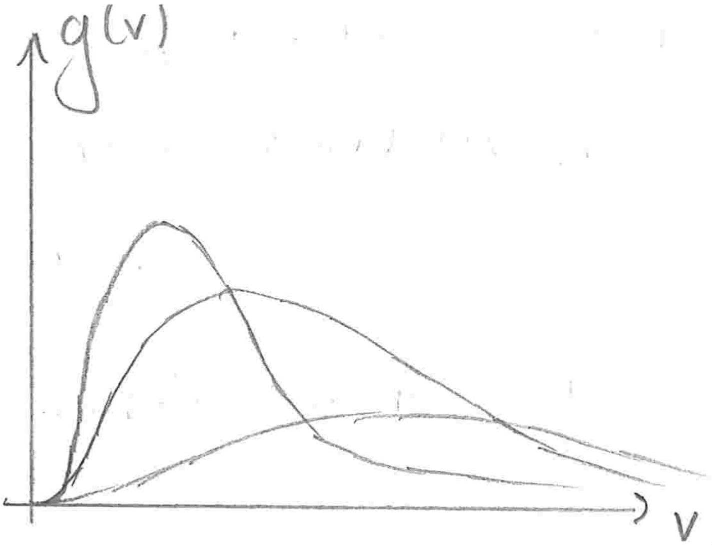
$$\Rightarrow f(v_x) = \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{m v_x^2}{2k_B T}}$$

and therefore

$$F(\vec{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{mv^2}{2k_B T}}.$$

and the speed distribution is just

$$g(v) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi v^2 e^{-\frac{mv^2}{2k_B T}}$$



Great. Let's move on to the hardcore stuff.

2. Dynamics of classical densities

Suppose we have an N-body system of particles labele[d] i with positions $\vec{q}_i$ and momenta $\vec{p}_i$. Any conservative dynamics can be described by a Hamiltonian $H(\vec{p}_i, \vec{q}_i)$ such that

$$\begin{cases} \dot{\vec{q}}_i = \dfrac{\partial H}{\partial \vec{p}_i} \\ \dot{\vec{p}}_i = -\dfrac{\partial H}{\partial \vec{q}_i} \end{cases}$$

The point $\mu(t) = (\vec{q}_i(t), \vec{p}_i(t))$ is said to live in the system's 6N dimensional phase-space, and flows around as time goes by.

If we know exactly the point $\mu(0)$, we can in principle integrate the equations of motion to get $\mu(t)$.

2.1. Liouville's theorem

Determining $\mu(0)$ for large systems, however, is an experimentally daunting task. We therefore resort to a probabilistic treatment. Define the phase space density $\rho(\mu, t)$ such that the probability to find the system in a small neighborhood $d\Gamma = \prod_i d^3\vec{q}_i\, d^3\vec{p}_i$ of μ is $\rho(\mu, t)\, d\Gamma$.

Then there are a few ways to realize that ρ behaves like a fluid moving through phase space,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial\vec{q}_i}\cdot\dot{\vec{q}}_i + \frac{\partial\rho}{\partial\vec{p}_i}\cdot\dot{\vec{p}}_i = \frac{\partial\rho}{\partial t} + \{\rho, H\} = 0.$$

The expectation value for time-independent observables $O(\mu)$ is given by

$$\langle O\rangle(t) = \int d\Gamma\, \rho(\mu, t)$$

with time evolution

$$\frac{d\langle O\rangle}{dt} = \langle\{O, H\}\rangle.$$

2.2. Statistical equilibrium

Equilibrium thermodynamics is concerned with a few macroscopic observables such as total energy U, temperature T, entropy S, pressure p...

How can we devise a distribution ρ_{eq} such that these observables don't depend on time?

An obvious way to do this is to set $\frac{\partial \rho_{eq}}{\partial t} = 0$.

Given a set of conserved quantities $\{L_\alpha\}$ for our system, this can easily be done by writing

$$\rho_{eq}(\mu) = \rho(H(\mu), L_\alpha(\mu)).$$

The dependence on H can be simply understood as the fundamental postulate of statistical mechanics.

2.3. Reversibility

Our present formalism cannot deal with the approach to equilibrium. Classical mechanics is reversible.

If we take $\mu(0) = (\vec{q}_i(0), \vec{p}_i(0))$

$\mapsto T\mu(0) = (\vec{q}_i(0), -\vec{p}_i(0))$, then we have

$$\mu(t) = (\vec{q}_i(t), \vec{p}_i(t))$$

$$T\mu(t) = (\vec{q}_i(-t), -\vec{p}_i(-t))$$

So for any state converging to ρ_{eq}, there's an equally good state diverging from it. We need a bit more depth in this description. Or maybe not? Let's think about it for a bit.

3. The BBGKY hierarchy

When we did baby kinetic theory, we inferred some macroscopic properties from the distribution of a single particle. Why exactly are we trying to describe the motion of 10^{23} of them?

That's a great question, and it's at the heart of what comes next. How many particles should we describe to be able to understand what's happening?

Suppose we have N indistinguishable particles in our system. Define the s-particle density

$$f(\vec{p}_1, \vec{q}_1, \dots, \vec{q}_s, t) = \frac{N!}{(N-s)!} \int \prod_{i=s+1}^{N} d\Gamma_i \, \rho(\vec{p}_1, \dots, \vec{q}_s, \vec{p}_{s+1}, \dots, \vec{q}_N, t)$$

as the expected number of s-tuplets of particles around a region of phase space.

The ideal gas Hamiltonian is

$$H = \sum_{i=1}^{N} \left[\frac{\vec{p}_i^{\,2}}{2m} + U(\vec{q}_i) \right] + \sum_{(i,j)=1}^{N} V(|\vec{q}_i - \vec{q}_j|)$$

To describe the evolution of f_s, it's convenient to break H into three parts

$$H_s = \sum_{n=1}^{s}\left[\frac{\vec{p}_n^{\,2}}{2m} + U(\vec{q}_n)\right] + \sum_{(n,m)=1}^{s} V(|\vec{q}_n - \vec{q}_m|)$$

$$H_{N-s} = \sum_{i=s+1}^{N}\left[\frac{\vec{p}_i^{\,2}}{2m} + U(\vec{q}_i)\right] + \sum_{(i,j)=s+1}^{N} V(|\vec{q}_i - \vec{q}_j|)$$

$$H' = \sum_{n=1}^{s}\sum_{i=s+1}^{N} V(|\vec{q}_n - \vec{q}_i|)$$

and we have

$$H = H_s + H_{N-s} + H'.$$

We now have by Liouville's theorem

$$\frac{\partial f_s}{\partial t} = \frac{N!}{(N-s)!}\int \prod_{i=s+1}^{N} d\Gamma_i \frac{\partial \rho}{\partial t}$$

$$= -\frac{N!}{(N-s)!}\int \prod_{i=s+1}^{N} d\Gamma_i \{\rho, H_s + H_{N-s} + H'\}$$

the first bracket is trivial,

$$\frac{\partial f_s}{\partial t} + \{f_s, H_s\} = -\frac{N!}{(N-s)!}\int \prod_{i=s+1}^{N} d\Gamma_i \{\rho, H_{N-s} + H'\}.$$

the second one gives 0,

$$\frac{\partial f_s}{\partial t} + \{f_s, H_s\} = -\frac{N!}{(N-s)!}\int \prod_{i=s+1}^{N} d\Gamma_i \{\rho, H'\}$$

The final one evaluates to

$$\frac{\partial f_s}{\partial t} + \{f_s, H_s\} = \sum_{n=1}^{s} \int d\Gamma_{s+1} \frac{\partial V(|\vec{q}_n - \vec{q}_{s+1}|)}{\partial \vec{q}_n} \cdot \frac{\partial f_{s+1}}{\partial \vec{p}_n} :$$

this is a hierarchy of N equations. We need to truncate it somewhere. But where?

4. The Boltzmann equation

Take the first two equations in the hierarchy.

$$\left[\frac{\partial}{\partial t} - \frac{\partial U}{\partial \vec{q}_1} \cdot \frac{\partial}{\partial \vec{p}_1} + \frac{\vec{p}_1}{m} \cdot \frac{\partial}{\partial \vec{q}_1}\right] f_1 = \int d\Gamma_2 \frac{\partial V(|\vec{q}_1 - \vec{q}_2|)}{\partial \vec{q}_1} \cdot \frac{\partial f_2}{\partial \vec{p}_1}$$

$$\left[\frac{\partial}{\partial t} - \frac{\partial U}{\partial \vec{q}_1} \cdot \frac{\partial}{\partial \vec{p}_1} - \frac{\partial U}{\partial \vec{q}_2} \cdot \frac{\partial}{\partial \vec{p}_2} + \frac{\vec{p}_1}{m} \cdot \frac{\partial}{\partial \vec{q}_1} + \frac{\vec{p}_2}{m} \cdot \frac{\partial}{\partial \vec{q}_2} - \frac{\partial V(|\vec{q}_1 - \vec{q}_2|)}{\partial \vec{q}_1} \cdot \left(\frac{\partial}{\partial \vec{p}_1} - \frac{\partial}{\partial \vec{p}_2}\right)\right] f_2 =$$

$$= \int d\Gamma_3 \left[\frac{\partial V(|\vec{q}_1 - \vec{q}_3|)}{\partial \vec{q}_1} \cdot \frac{\partial}{\partial \vec{p}_1} + \frac{\partial V(|\vec{q}_2 - \vec{q}_3|)}{\partial \vec{q}_2} \cdot \frac{\partial}{\partial \vec{p}_2}\right] f_3$$

We have three relevant time scales.

(a) $\frac{1}{\tau_U} \sim \frac{\partial U}{\partial \vec{q}} \cdot \frac{\partial}{\partial \vec{p}} \longrightarrow$ external interactions

(b) $\frac{1}{\tau_c} \sim \frac{\partial V}{\partial \vec{q}} \cdot \frac{\partial}{\partial \vec{p}} \longrightarrow$ collision durations

(c) $\frac{1}{\tau_x} \sim \int d\Gamma \frac{\partial V}{\partial \vec{q}} \cdot \frac{\partial}{\partial \vec{p}} \frac{f_{s+1}}{f_s} \longrightarrow$ survival time

If the particles have number density n and a characteristic volume d^3, we can write

$$\frac{1}{\tau_x} \sim \frac{n d^3}{\tau_c}$$

So if the particles are very dilute, the right hand sides grow smaller and smaller!
Let's truncate the hierarchy setting the second r.h.s. to zero.
Now note that the equation for f_2 has collision terms. It should evolve much faster than f_1, so we can approximate it to a "steady state". Also, the external terms are not at all relevant during a collision.
With these considerations, the equation for f_2 is

$$\left[\frac{\vec{p}_1}{m} \cdot \frac{\partial}{\partial \vec{q}_1} + \frac{\vec{p}_2}{m} \cdot \frac{\partial}{\partial \vec{q}_2} - \frac{\partial V(|\vec{q}_1 - \vec{q}_2|)}{\partial \vec{q}_1} \cdot \left(\frac{\partial}{\partial \vec{p}_1} - \frac{\partial}{\partial \vec{p}_2} \right) \right] f_2 = 0.$$

For similar reasons, f_2 should be much more sensitive to variations of the relative coordinate $\vec{q} = \vec{q}_2 - \vec{q}_1$ than of the C.M. $\vec{Q} = \frac{\vec{q}_1 + \vec{q}_2}{2}$. We then write

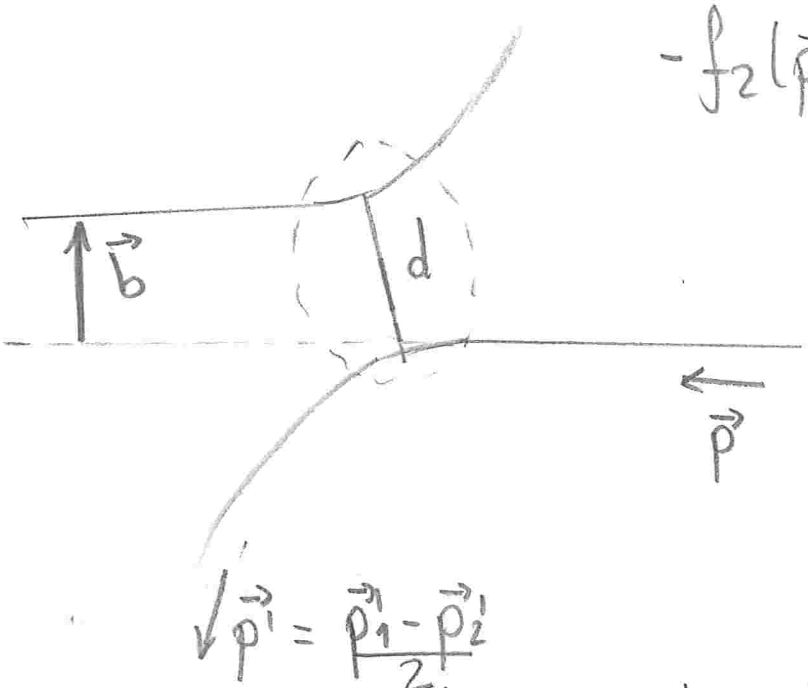
$$\frac{\partial V(|\vec{q}_1 - \vec{q}_2|)}{\partial \vec{q}_1} \cdot \left(\frac{\partial}{\partial \vec{p}_1} - \frac{\partial}{\partial \vec{p}_2} \right) f_2 \approx -\left(\frac{\vec{p}_1 - \vec{p}_2}{m} \right) \cdot \frac{\partial f_2}{\partial \vec{q}}.$$

Integrating both sides of this equation gives the R.H.S of the equation for f_1,

$$\underbrace{\int d\Gamma_2 \frac{\partial V(|\vec{q_1}-\vec{q_2}|)}{\partial \vec{q_1}} \cdot \frac{\partial f_2}{\partial \vec{p_1}}}_{\dot{f}_1|_{coll}} \approx \int d^3\vec{p_2}\, d^3\vec{q} \left(\frac{\vec{p_2}-\vec{p_1}}{m}\right) \cdot \frac{\partial}{\partial \vec{q}} f_2(\vec{p_1}, \vec{q_1}, \vec{p_2}, \vec{q_1}+\vec{q}, t).$$

We integrate on $\vec{q}$ along the direction of $\vec{p} = \frac{\vec{p_1}-\vec{p_2}}{2}$, giving

$$\int d\Gamma_2 \frac{\partial V(|\vec{q_1}-\vec{q_2}|)}{\partial \vec{q_1}} \cdot \frac{\partial f_2}{\partial \vec{p_2}} \approx \int d^3\vec{p_2}\, d^2\vec{b}\, |\vec{v_1}-\vec{v_2}| \left[f_2(\vec{p_1}, \vec{q_1}, \vec{p_2}, \vec{b}, +, t) - f_2(\vec{p_1}, \vec{q_1}, \vec{p_2}, \vec{b}, -, t)\right]$$



$$\vec{p_1}' = \frac{\vec{p_1}+\vec{p_2}+|\vec{p_1}-\vec{p_2}|\hat{\Omega}}{2}$$

$$\vec{p_2}' = \frac{\vec{p_1}+\vec{p_2}-|\vec{p_1}-\vec{p_2}|\hat{\Omega}}{2}$$

by time-reversal symmetry, $f_2(\vec{p_1}, \vec{q_1}, \vec{p_2}, \vec{b}, +, t) = f_2(\vec{p_1}', \vec{q_1}, \vec{p_2}', \vec{b}, -, t)$.

we substitute and get

$$\int d\Gamma_2 \frac{\partial V(|\vec{q}_1-\vec{q}_2|)}{\partial \vec{q}_1} \cdot \frac{\partial f_2}{\partial \vec{p}_1} = \int d^3\vec{p}_2\, d^2\vec{b}\, |\vec{v}_1-\vec{v}_2| \left[f_2(\vec{p}_1{}', \vec{q}_1, \vec{p}_2{}', \vec{b}, -, t) - f_2(\vec{p}_1, \vec{q}_1, \vec{p}_2, \vec{b}, -, t) \right]$$

we can also write this in terms of the differential cross-section as

$$\int d\Gamma_2 \frac{\partial V(|\vec{q}_1-\vec{q}_2|)}{\partial \vec{q}_1} \cdot \frac{\partial f_2}{\partial \vec{p}_1} \approx \int d^3\vec{p}_2\, d^2\Omega \left| \frac{d\sigma}{d\Omega} \right| |\vec{v}_1-\vec{v}_2| \left[f_2(\vec{p}_1{}', \vec{q}_1, \vec{p}_2{}', \vec{b}, -, t) - f_2(\vec{p}_1, \vec{q}_1, \vec{p}_2, \vec{b}, -, t) \right]$$

We now use the assumption of molecular chaos,

$$f_2(\vec{p}_1, \vec{q}_1, \vec{p}_2, \vec{b}, -, t) \approx f_1(\vec{p}_1, \vec{q}_1, t) f_1(\vec{p}_2, \vec{q}_1, t),$$

and get the Boltzmann equation,

$$\left[\frac{\partial}{\partial t} - \frac{\partial U}{\partial \vec{q}_1} \cdot \frac{\partial}{\partial \vec{p}_1} + \frac{\vec{p}_1}{m} \cdot \frac{\partial}{\partial \vec{q}_1} \right] f_1 = - \int d^3\vec{p}_2\, d^2\Omega \left| \frac{d\sigma}{d\Omega} \right| |\vec{v}_1-\vec{v}_2| \left[f_1(\vec{p}_1, \vec{q}_1, t) f_1(\vec{p}_2, \vec{q}_1, t) - f_1(\vec{p}_1{}', \vec{q}_1, t) f_1(\vec{p}_2{}', \vec{q}_1, t) \right].$$

5. The H-theorem

$$H(t) = \int d^3\vec{p}\, d^3\vec{q}\; f_1 \log f_1$$

$$\dot{H} = \int d^3\vec{p}\, d^3\vec{q}\; \log f_1 \frac{\partial f_1}{\partial t}$$

$$= \int d^3\vec{p}\, d^3\vec{q}\; \log f_1 \left(\frac{\partial U}{\partial \vec{q}} \cdot \frac{\partial f_1}{\partial \vec{p}} - \frac{\vec{p}}{m} \cdot \frac{\partial f_1}{\partial \vec{q}} \right)$$

$$- \int d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{q}\, d^2\vec{b}\; |\vec{v}_1 - \vec{v}_2| \log f_1(\vec{p}_1, \vec{q})$$

$$\left[f_1(\vec{p}_1, \vec{q}) f_1(\vec{p}_2, \vec{q}) - f_1(\vec{p}_1', \vec{q}) f_1(\vec{p}_2', \vec{q}) \right]$$

The first term vanishes, and in the second we can swap $\vec{p}_1 \leftrightarrow \vec{p}_2$

$$\dot{H} = -\frac{1}{2} \int d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{q}\, d^2\vec{b}\; |\vec{v}_1 - \vec{v}_2| \log f_1(\vec{p}_1, \vec{q}) f_1(\vec{p}_2, \vec{q})$$

$$\left[f_1(\vec{p}_1, \vec{q}) f_1(\vec{p}_2, \vec{q}) - f_1(\vec{p}_1', \vec{q}) f_1(\vec{p}_2', \vec{q}) \right]$$

Doing something similar for $\vec{p} \leftrightarrow \vec{p}'$,

$$\dot{H} = -\frac{1}{4} \int d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{q}\, d^2\vec{b}\; |\vec{v}_1 - \vec{v}_2|$$

$$\left[f_1(\vec{p}_1) f_1(\vec{p}_2) - f_1(\vec{p}_1') f_2(\vec{p}_2') \right]$$

$$\left[\log f_1(\vec{p}_1) f_1(\vec{p}_2) - \log f_1(\vec{p}_1') f_1(\vec{p}_2') \right] \leq 0.$$

6. Local equilibrium

Setting the integrand to zero,

$$f_1(\vec{p}_1) f_1(\vec{p}_2) = f_1(\vec{p}_1') f_1(\vec{p}_2')$$

so $\log f_1$ is a linear combination of any additive conserved quantity for a collision, and by comparison with the M-B distribution, we write the local equilibrium distribution

$$f_1^{(0)}(\vec{p},\vec{q},t) = \frac{n(\vec{q},t)}{(2\pi m k_B T(\vec{q},t))^{3/2}} \exp\left[-\frac{(\vec{p}-m\vec{v}(\vec{q},t))^2}{2m k_B T(\vec{q},t)}\right]$$

where n is the local particle number density,

$$n(\vec{q},t) = \int d^3\vec{p}\, f_1(\vec{p},\vec{q},t)$$

and we have local expectation values for any single particle operator $O(\vec{p},\vec{q})$

$$\langle O\rangle(\vec{q},t) = \frac{1}{n(\vec{q},t)} \int d^3\vec{p}\, O(\vec{p},\vec{q}) f_1(\vec{p},\vec{q},t).$$

7. Conservation Laws

Let's address how exactly f_1 can approach an equilibrium distribution.

(i) $f_2(\vec{q}_1, \vec{q}_2)$ relaxes to $f_1(\vec{q}_1) f_1(\vec{q}_2)$ at times $\sim \tau_c$

(ii) f_1 relaxes to local equilibrium locally maximizing H at every point, at time scales τ_x

(iii) f_1 relaxes to global equilibrium at time scales τ_u.

Consider now a single particle quantity χ that is additively conserved in a collision,

$$\chi(\vec{p}_1, \vec{q}, t) + \chi(\vec{p}_2, \vec{q}, t) = \chi(\vec{p}_1{}', \vec{q}, t) + \chi(\vec{p}_2{}', \vec{q}, t).$$

Then we have

$$J\chi(\vec{q}, t) = \int d^3p \, \chi(\vec{p}, \vec{q}, t) \, \dot{f}_1|_{coll}(\vec{p}, \vec{q}, t) = 0.$$

Using then the Boltzmann equation we get

$$J\chi = \int d^3\vec{p} \, \chi(\vec{p}, \vec{q}, t) \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{q}} + \vec{F} \cdot \frac{\partial}{\partial \vec{p}} \right] f_1(\vec{p}, \vec{q}, t)$$

where $\vec{F} = -\frac{\partial U}{\partial \vec{q}}$ is the external force. Using product rules we get

$$\frac{\partial}{\partial t}(n\langle\chi\rangle)+\frac{\partial}{\partial\vec{q}}\cdot\left(n\left\langle\frac{\vec{p}}{m}\chi\right\rangle\right)-n\left\langle\frac{\partial\chi}{\partial t}\right\rangle$$

$$-n\left\langle\frac{\vec{p}}{m}\cdot\frac{\partial\chi}{\partial\vec{q}}\right\rangle-n\vec{F}\cdot\left\langle\frac{\partial\chi}{\partial\vec{p}}\right\rangle=0$$

Let's substitute some χ's and see what we get

(a) Particle number:

If $\chi=1$,

$$\partial_t n+\partial_i(nv_i)=0$$

where we've introduced $\vec{v}=\langle\vec{p}/m\rangle$ the local average velocity

(b) Momentum:

If $\chi=\vec{c}=\frac{\vec{p}}{m}-\vec{v}$, we get

$$nm(\partial_t+v_j\partial_j)v_i=-\partial_j P_{ij}-nF_i$$

where we've defined the pressure tensor $P_{ij}=nm\langle c_ic_j\rangle$

(c) Kinetic energy

If $\chi=\frac{1}{2}mc^2$, using the continuity, we get

$$\partial_t\varepsilon+v_i\partial_i\varepsilon=-\frac{1}{n}\partial_i h_i-\frac{1}{n}P_{ij}v_{ij}$$

where we've introduced the average energy

$$\varepsilon = \langle \frac{1}{2} m c^2 \rangle$$

the heat flux

$$\vec{h} = \frac{nm}{2} \langle \vec{c} c^2 \rangle$$

and the strain rate

$$v_{ij} = \frac{1}{2} (\partial_i v_j + \partial_j v_i)$$

So given expressions for P_{ij} and $\vec{h}$, these equations can in principle be solved. These are Navier-Stokes.

8. Zero-order hydrodynamics

If we use $f_1^{(0)}$ as an estimate for the density in a fluid, we get in particular

$$P_{ij}^{(0)} = n k_B T \delta_{ij}, \quad \vec{h}^{(0)} = 0, \quad \varepsilon^{(0)} = \frac{3}{2} k_B T$$

which leads to the fluid equations

$$\partial_t n + \nabla \cdot (n \vec{v}) = 0$$

$$nm(\partial_t + \vec{v} \cdot \nabla) \vec{v} = -\nabla (n k_B T) + n\vec{F}$$

$$(\partial_t + \vec{v} \cdot \nabla) T = -\frac{2}{3} T \nabla \cdot \vec{v}.$$

and with the first and third equations we get

$$(\partial_t + \vec{v}\cdot\nabla)\left(\frac{n}{T^{3/2}}\right)=0,$$

i.e. the entropy density is constant along streamlines. These equations then describe adiabatic flow.

9. First-order hydrodynamics

The local equilibrium solution sets the collision terms of the Boltzmann equation to zero. The streaming terms, however, aren't zero. Acting on $f_1^{(0)}$ with them we get

$$\left[D_t + \vec{c}\cdot\nabla + \frac{\vec{E}}{m}\cdot\frac{\partial}{\partial\vec{c}}\right]\log f_1^{(0)} = \mathcal{L}[\log f_1^{(0)}]$$

and if the fields satisfy the hydrodynamic equations of the previous section we get

$$\mathcal{L}[\log f_1^{(0)}] = \frac{m}{k_B T}\left(c_i c_j - \frac{\delta_{ij}}{3}c^2\right)v_{ij} + \left(\frac{mc^2}{2k_B T}-\frac{5}{2}\right)\frac{c_i}{T}\partial_i T.$$

We now estimate the r.h.s. of the Boltzmann eq. as follows. Expand f_1 in a collision parameter $g(\vec{p})$ as $f_1^{(1)}\approx f_1^{(0)}(1+g)$. Then we get

$$\dot{f}_1\big|_{coll} \approx -\int d^3p_2\, d^2\vec{b}\, |\vec{v}_1-\vec{v}_2|\, f_1^{(0)}(\vec{p}_1) f_1^{(0)}(\vec{p}_2)$$
$$\left[g(\vec{p}_1)+g(\vec{p}_2)-g(\vec{p}_1{}')-g(\vec{p}_2{}')\right]$$

$$f_1|_{coll} \approx - f_1^{(0)} \frac{g}{\tau_x}.$$

Then we get

$$g = -\tau_x \mathcal{L}[\log f_1^{(0)}]$$

So the correction to f_1 to leading order in τ_x is

$$f_1^{(1)} \approx f_1^{(0)} \Big[1 - \frac{\tau_\mu m}{k_B T}\Big(c_i c_j - \frac{\delta_{ij} c^2}{3}\Big) u_{ij} - \tau_K \Big(\frac{mc^2}{2k_B T} - \frac{5}{2}\Big) \frac{c_i}{T} \partial_i T \Big].$$

In this scheme, we get

$$P_{ij}^{(1)} = n k_B T - 2 n k_B T \tau_\mu \Big(u_{ij} - \frac{\delta_{ij}}{3} u_{kk}\Big)$$

$$\vec{h}^{(1)} = -\frac{5}{2} \frac{n k_B^2 T \tau_K}{m} \nabla T$$

(a) We get viscosity with (off-diagonal)

$$\mu = 2 n k_B T \tau_\mu$$

(b) We get Fourier's Law with

$$K = \frac{5}{2} \frac{n k_B^2 T \tau_K}{m}$$