And then We said: Let there be Light!

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Abstract

Constructing physical theories from the ground up is usually quite a daunting task. Choosing equations to describe phenomena typically relies on arbitrarily large sets of experimental data and many hours of educated (or not) guessing to eventually reproduce results. But what if we went the other way around? In this lecture we shall consider a set of reasonable and relatively weak assumptions. After a few calculations regarding their consequences we will naturally arrive at Maxwell's equations for electrodynamics.

1. Introduction

The well known Maxwell's equations encode the main experimental facts about electromagnetic fields: Coulomb's Law, the absence of magnetic monopoles, Faraday's Law, and the Ampère-Maxwell Law. In natural units $\varepsilon_0 = \mu_0 = 1$, they are

$$\begin{aligned} \nabla \cdot \vec{E} &= \rho \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\partial_t \vec{B} \\ \nabla \times \vec{B} &= \vec{j} + \partial_t \vec{E} \end{aligned} \qquad (1)$$

where ρ and $\vec{j}$ are the electric charge and current density, and the electromagnetic fields $\vec{E}$ and $\vec{B}$ are defined by the force $\vec{F}$ experienced by a particle of charge q moving with velocity $\vec{v}$ through the Lorentz Force Law:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{2}$$

Note in particular that the equations imply the conservation of electric charge,

$$\partial_t \rho + \nabla \cdot \vec{j} = 0, \tag{3}$$

as well as indicate there is a repulsive force between stationary point charges of equal sign

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi r_{12}^2} \hat{r}_{12}. \tag{4}$$

2. Assumptions

* Special Relativity applies (Poincaré Group symmetries)
* Electric charge Q exists, can be continuously distributed as a density ρ, and is conserved

$$\begin{gathered} Q = \int d^3x \, \rho, \\ j = (\rho, \vec{j}), \\ \partial_\mu j^\mu = 0. \end{gathered} \tag{5}$$

2 * There are no magnetic charges

* Point charges interact locally and weakly with each other and obey superposition

$$f^{\mu} = q F^{\mu}{}_{\nu} V^{\nu}. \tag{6}$$

* Particles infinitely far away don't interact

3. The Force Law

Let's examine the force law. Suppose our charged particle has some finite mass m. Its momentum $p = mv$ then satisfies

$$p_{\mu} p^{\mu} = m^2.$$

The force is defined as

$$f = \frac{dp}{d\tau}$$

$$\frac{d}{d\tau}(p_{\mu} p^{\mu}) = f_{\mu} p^{\mu} + p_{\mu} f^{\mu} = 0$$

$$\Rightarrow g_{\mu\nu} f^{\mu} p^{\nu} = 0$$

$$g_{\mu\nu} q F^{\mu}{}_{\sigma} V^{\sigma} m V^{\nu} = 0$$

$$m q F_{\mu\nu} V^{\mu} V^{\nu} = 0.$$

We conclude that $F_{\mu\nu}$ is antisymmetric. Immediately this implies the Lorentz Force Law,

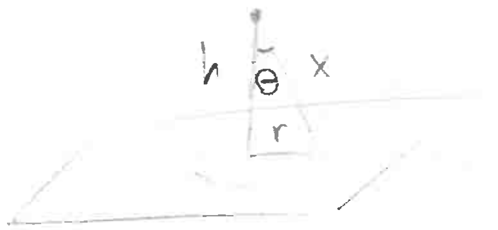
$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}),$$

with
$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_x & -B_y & B_x & 0 \end{pmatrix},$$
which is just the Faraday tensor. We are going very far very quickly! But we still need all the equations of motion for these force fields. Let's try to get them now.

4. Coulomb's Law

We're going to study the force between stationary point charges. Suppose they have charges q_1 and q_2 at positions $\vec{r}_1$ and $\vec{r}_2$. Rotational and translational symmetry imply that the forces are such that
$$\vec{F}_{12} = -\vec{F}_{21}$$
$$\vec{F}_{12} \times (\vec{r}_1 - \vec{r}_2) = 0.$$
The linearity in charge then implies, together with translational symmetry, that
$$\vec{F}_{12} = q_1 q_2 f(\|\vec{r}_1 - \vec{r}_2\|) \frac{\vec{r}_1 - \vec{r}_2}{\|\vec{r}_1 - \vec{r}_2\|},$$
where f is to be determined. We can define (due to the non interaction of far away charges) a potential function
$$u(r) = \int_r^\infty f(x)\,dx.$$
Let's then use these unknown functions and compute the electic field for a particular pair of distributions.

4.1. The infinite plane



The electric field of an infinite plane of charge density σ is computed as usual.

$$dq = 2\pi r\, dr\, \sigma$$

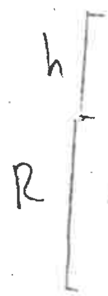
$$dE = dq\, f(x) \cos\theta$$

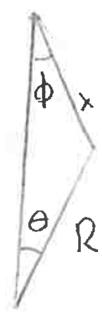
$$dE = 2\pi\sigma h f(x)\, dx$$

$$E_{plane} = 2\pi\sigma h \int_h^\infty f(x)\, dx$$

$$\boxed{E_{plane} = 2\pi\sigma h\, u(h)} \qquad (7)$$

4.2. A very special spherical cap.





Choose a point and a sphere of radius R a distance h from it. Fill the surface within the upper part of the tangent cone with charge density σ.

$$dq = 2\pi R^2 \sigma \sin\theta\, d\theta$$

$$dE = dq\, f(x) \cos\phi$$

$$dE = 2\pi R^2 \sigma f(x) \sin\theta \cos\phi\, d\phi$$

$$\sin\theta\, d\theta = \frac{x\, dx}{R(R+h)}$$

$$\cos\phi = \frac{x^2+2Rh+h^2}{2(R+h)x}$$

$$dE = 2\pi\sigma \frac{R}{R+h} f(x) \frac{x^2+2Rh+h^2}{2(R+h)} dx$$

$$E_{cap} = 2\pi\sigma \frac{R}{R+h} \int_h^{\sqrt{2Rh+h^2}} f(x) \frac{x^2+2Rh+h^2}{2(R+h)} dx$$

$$E_{cap} = 2\pi\sigma h \frac{R}{R+h} \left[u(h) - \frac{2R+h}{R+h} u(\sqrt{2Rh+h^2}) + \frac{1}{h(R+h)} \int_h^{\sqrt{2Rh+h^2}} x u(x) dx \right] \tag{8}$$

4.3. A sneaky argument

We now expect that $E_{cap} \approx E_{plane}$ when $R \gg h$. If we define $y = \sqrt{2Rh+h^2}$, this is equivalent to

$$\frac{1}{y^2} \int_h^y x u(x) dx \approx u(y), \quad y \gg h$$

Differentiating both sides of the equation we finally get

$$u(x) = \frac{K}{x}.$$

We want repulsion, so we choose $K = 1/4\pi$, and finally deduce Coulomb's Law:

$$f(r) = \frac{1}{4\pi r^2}.$$

5. The Biot-Savart law

In this section we shall derive the magnetic field of slowly moving charges

5.1. The magnetic field of a stationary charge

Put a charge q fixed at the origin. From rotational symmetry, its field should be $\vec{B}(\vec{r}) = B(r)\,\hat{r}$. Now let's get the force it does on a moving charge with velocity $\vec{v} \perp \vec{r}$.

Let $\vec{r} = R\hat{y}$, $\vec{v} = v\hat{x}$. Then the total force is

$$\vec{F} = q'\left[\frac{q}{4\pi R^2}\hat{y} + (v\hat{x}) \times (B(R)\hat{y})\right]$$

$$= \frac{q'q}{4\pi R^2}\hat{y} + vB(R)\hat{z}.$$

This has a component perpendicular to the plane of interaction. It violates reflection symmetry $\Rightarrow B(R) = 0$.

5.2. The field of a slow charge

Suppose now we have a charge q moving with velocity $\vec{v}$ at the origin when $t=0$. Let's now measure the force it does on a charge q' with position $\vec{r}(t) = \vec{R} + \vec{v}t$.

We write down the electromagnetic field of this point charge as $\vec{E}(\vec{r}), \vec{B}(\vec{r})$. When we do a Lorentz transformation with velocity $\vec{v}$, the field transforms as

$$\vec{E}'_{\parallel}(x') = \vec{E}_{\parallel}(x)$$

$$\vec{B}'_{\parallel}(x') = \vec{B}_{\parallel}(x)$$

$$\vec{E}'_{\perp}(x') = \gamma\left(\vec{E}_{\perp}(x) + \vec{v} \times \vec{B}_{\perp}(x)\right)$$

$$\vec{B}'_{\perp}(x') = \gamma\left(\vec{B}_{\perp}(x) - v \times \vec{E}_{\perp}(x)\right)$$

But after the transformation,

$$\vec{E}'(\vec{r}') = \frac{q}{4\pi r'^2}\hat{r}'$$

$$\vec{B}(\vec{r}') = 0$$

Then ($\gamma \approx 1$)
$$\vec{E}_{\parallel}(t,\vec{r}) = \frac{(\vec{E}'(\vec{r}') \cdot \vec{v})\vec{v}}{v^2}$$

$$\vec{B}_{\parallel}(t,\vec{r}) = 0$$

$$\vec{B}_{\perp}(t,\vec{r}) = \vec{v} \times \vec{E}_{\perp}(t,\vec{r})$$

$$\vec{E}_{\perp}(t,\vec{r}) + \vec{v} \times \vec{B}_{\perp}(t,\vec{r}) = \vec{E}'(\vec{r}) - \frac{(\vec{E}'(\vec{r}') \cdot \vec{v})\vec{v}}{v^2}$$

These equations then imply that

$$\vec{B}(t,\vec{r}) = \vec{v} \times \vec{E}'(\vec{r}')$$

If we now remember how to write $\vec{r}'$ in terms of $\vec{r}$ and t:

$$\left.\begin{array}{l} \vec{r}'_{\perp} = \vec{r}_{\perp} \\ \vec{r}'_{\parallel} = \gamma(\vec{r}_{\parallel} - \vec{v}t) \end{array}\right\} \Rightarrow \vec{r}' \approx \vec{r} - \vec{v}t$$

But we are only interested in $t=0$, where the charge is at the origin, so $\vec{r}' \approx \vec{r}$

$$\Rightarrow \vec{B}(0,\vec{r}) = q\frac{\vec{v} \times \hat{r}}{4\pi r^2}. \qquad (9)$$

This is the Biot-Savart Law.

6. Gauss' and Ampère' Law

Slowly moving is equivalent to stationary, as it has all to do with causality. As such, we got that the electromagnetic field of a stationary distribution $(\rho, \vec{j})$ should be

$$\vec{E}(\vec{r}) = \int d^3r' \frac{1}{4\pi} \rho(\vec{r}') \frac{\vec{r}-\vec{r}'}{\|\vec{r}-\vec{r}'\|^3}$$

$$\vec{B}(\vec{r}) = \int d^3r' \frac{1}{4\pi} \vec{j}(\vec{r}') \times \frac{\vec{r}-\vec{r}'}{\|\vec{r}-\vec{r}'\|^3}$$

These then imply

$$\begin{aligned} \nabla \cdot \vec{E}(\vec{r}) &= \rho(\vec{r}) \\ \nabla \times \vec{B}(\vec{r}) &= \vec{j}(\vec{r}). \end{aligned} \tag{10}$$

These are the Gauss and Ampère Laws.

7. Relativity to the max!

Now we are going to use the expression (5) to try to generalize (10) into (11). To do so, we introduce the following theorem

Theorem (Amazing Greens function stuff):
Let J^μ be a divergenceless vector field. Then the antisymmetric tensor field

$$F^{\mu\nu}(x) = \int d^4x' \, G(x,x') \left(\partial'^\mu J^\nu(x') - \partial'^\nu J^\mu(x') \right),$$

where $G(x,x') = \frac{\delta(t'-t+R)}{4\pi R}$, $R = \|\vec{x}-\vec{x}'\|$ is the retarded propagator of the wave equation, satisfies the relations

$$\partial_\mu \mathcal{F}^{\mu\nu} = j^\nu, \quad \partial_\mu {}^*\mathcal{F}^{\mu\nu} = 0,$$

with ${}^*\mathcal{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\mathcal{F}_{\alpha\beta}$, the dual tensor field.

Proof:

Start with

$$\partial'_\mu j^\mu(x') = 0.$$

Apply $-G(x,x')\partial'^\nu$ and add $j^\nu \partial'_\mu \partial'^\mu G(x,x')$

$$-G(x,x')\partial'^\nu \partial'_\mu j^\mu(x') + j^\nu(x')\partial'_\mu \partial'^\mu G(x,x') = j^\nu \partial'_\mu \partial'^\mu G(x,x')$$

The LHS version of the term can be written as

$$-\partial'_\mu G(\partial'^\mu j^\nu - \partial'^\nu j^\mu) - \partial'_\mu(G\partial'^\nu j^\mu - j^\nu \partial'^\mu G)$$

and then using that $\partial_\mu G = -\partial'_\mu G$, as well as $\partial'_\mu \partial'^\mu G = \delta(x-x')$, we get that

$$\partial_\mu G(\partial'^\mu j^\nu - \partial'^\nu j^\mu) = j^\nu \delta(x-x') + \partial'_\mu(G\partial'^\nu j^\mu - j^\nu \partial'^\mu G)$$

then

$$\partial_\mu \mathcal{F}^{\mu\nu} = j^\nu + \int d^4x' \, \partial'_\mu(G\partial'^\nu j^\mu - j^\nu \partial'^\mu G)$$

$$\partial_\mu \mathcal{F}^{\mu\nu} = j^\nu.$$

Now to prove that it works for the dual, note that

$$*\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} \int d^4x' \, G \partial'_\alpha j^\beta$$

Integrate by parts to get

$$*\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} \int d^4x \, \partial'_\alpha (G j^\beta) - \partial'_\alpha G j^\beta$$

$$= -\varepsilon^{\mu\nu\alpha\beta} \int d^4x \, \partial'_\alpha G j^\beta$$

Now apply the derivative

$$\partial_\mu *\tilde{F}^{\mu\nu} = -\varepsilon^{\mu\nu\alpha\beta} \int d^4x \, \partial_\mu \partial'_\alpha G j^\beta$$

$$= 0. \qquad \square$$

We have proven a very interesting result.

8. Maxwell's equations.

If we remind ourselves of $F^{\mu\nu}$ for the theory we're constructing

$$F^{i0} = E_i,$$

$$F^{ij} = -\varepsilon^{ijk} B_k,$$

we see that equations (10) are simply

$$\partial_\mu F^{\mu\nu} = j^\nu.$$

Since a stationary case is just a special condition of the general case, this begs us to use the theorem we got and conclude that the equations of motion for the electromagnetic field must be

$$\partial_\mu F^{\mu\nu} = j^\nu$$
$$\partial_\mu {}^*F^{\mu\nu} = 0. \qquad (11)$$

But why this second equation? Well, note that

$${}^*F^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix},$$

which is pretty much taking the theory and swapping $\vec{E} \leftrightarrow \vec{B}$. As such, having an inhomogeneous term in the second equation would be equivalent to having some other type of charge that creates magnetic fields. But we have thrown out that possibility already!

Expanding the equations of motion we get Maxwell's equations

$$\nabla \cdot \vec{E} = \rho$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\partial_t \vec{B}$$
$$\nabla \times \vec{B} = \vec{j} + \partial_t \vec{E}$$

Amazing!

9. Let there be light!

9.1. Gauge invariance

Take the homogeneous equation. It can be trivially solved by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

for any vector field A, called a vector potential.

And so we can encode the entire theory in terms of A:

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = j^\nu. \qquad (12)$$

This is nice. But there's a cool quirk in this formalism. Take a scalar Λ. If we transform the fields

$$A^\mu \to A^\mu + \partial^\mu \Lambda,$$

then $F^{\mu\nu}$ is unchanged. As such, we can choose Λ freely and still describe the same physics! This is called gauge freedom, and in the Lagrangian formulation

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

it is a symmetry that gives back charge conservation.

9.2. The wave equation

Now that we know about gauges, rewrite (12)

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = j^\nu.$$

In free space, $j = 0$ so

$$\partial_\mu \partial^\mu A^\nu = \partial^\nu (\partial_\mu A^\mu).$$

This is almost a wave equation! It would be great if we could make the right hand side go to zero. Let's use gauges. Suppose a field A^μ solves the equations of motion. Can we find Λ such that $\partial_\mu (A^\mu + \partial^\mu \Lambda) = 0$? Well, of course!

$$\partial_\mu \partial^\mu \Lambda = -\partial_\mu A^\mu$$

is just the scalar wave equation with a source. It always has solutions for well behaved A^μ.

Finally, we can impose the Lorenz gauge (yes, without the t)

$$\partial_\mu A^\mu = 0$$

and get a relativistic wave equation for the vector potential

$$\boxed{\partial_\mu \partial^\mu A^\nu = 0}$$

we have light!