The Geometry of Spacetime

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March 28, 2018

A little bit of mechanics

Nature makes the action

$$S = \int_{t_a}^{t_b} L \, \mathrm{d}t$$

as small as possible

A little bit of mechanics

This happens when $L = T - V$ satisfies

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i}$$

These are the so called Euler-Lagrange equations

For a free particle it's just a line

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

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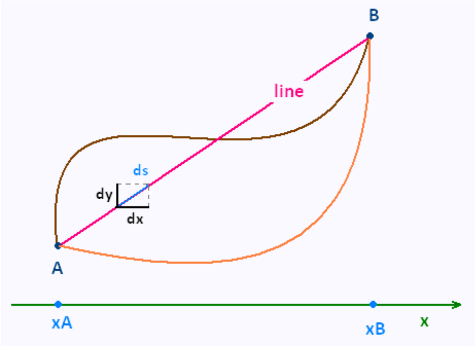

Figure: Motion of a free particle

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$$v^2 = R^2\dot{\theta}^2 + R^2\dot{\phi}^2 \sin^2\theta$$

Which gives us ...

What about surfaces?

$$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$
$$\ddot{\phi} + 2\cot\theta\,\dot{\theta}\,\dot{\phi} = 0$$

Very complicated... let's use symmetry !

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Particle in the equator with constant angular velocity is a solution

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Symmetry says that every Great Circle is a solution !

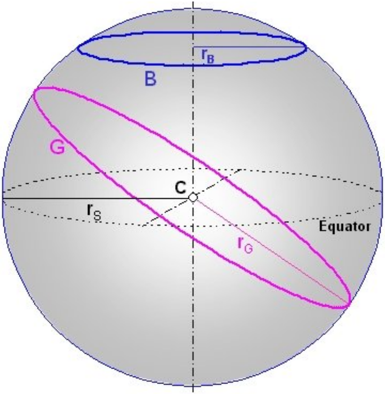

Figure: Great circle in purple $\mathcal{S}^2$

Free particles moves in geodesics

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For a free particle in $\mathcal{S}^2$, where $x^1 = \theta$ and $x^2 = \phi$:

$$\ddot{x}^1 + \left(-\sin x^1 \cos x^1\right) \dot{x}^2 \dot{x}^2 = 0$$
$$\ddot{x}^2 + \left(2 \cot x^1\right) \dot{x}^2 \dot{x}^2 = 0$$

Geodesic equation for a general surface embedded in $\mathbb{R}^3$:

$$\ddot{x}^i + \sum_{j,k} \Gamma^i{}_{jk} \dot{x}^j \dot{x}^k = 0$$

Information about the curvature must be encoded in the numbers $\Gamma^i{}_{jk}$, called the Christoffel symbols

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From now on we use the Einstein convention, so the equation above becomes:

$$\ddot{x}^i + \Gamma^i{}_{jk} \dot{x}^j \dot{x}^k = 0$$

How to bend spacetime

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Geodesic equations in a curved spacetime!

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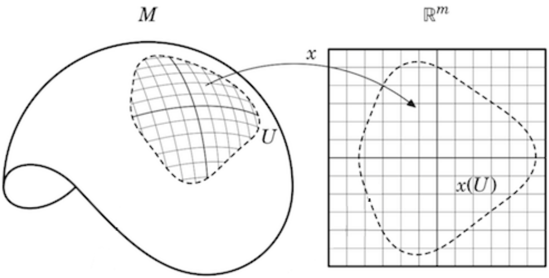

Figure: Spacetime may be something crazy

Topological Spaces

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The pair (M, τ) is called a *topological space*. Elements of τ are called open sets.

Continuous maps, Homeomorphism

A map $f : (X, \tau_X) \to (Y, \tau_Y)$ is said to be continuous if:

$$\text{for every } V \in \tau_Y \,, \; f^{-1}(V) \in \tau_X$$

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A bijection $X \stackrel{\varphi}{\longleftrightarrow} Y$ is said to be a *homeomorphism* if it's continuous both ways.
We say that X and Y are *homeomorphic* if there's a homeomorphism between them.

Charts

Let (M, τ) be a topological space. A chart in M is a pair (U, x) where $U \in \tau$ and $x : U \to \mathbb{R}^n$ is a homeomorphism.

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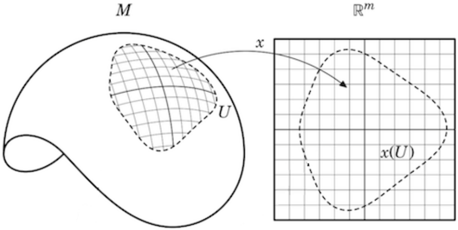

Figure: I've seen this before...

Smooth Atlas

A smooth atlas $\mathcal{A}$ in a topological space (M, τ) is a collection of charts $\mathcal{A} = \{(U_\alpha, x_\alpha)\}$ such that:

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  $$y \circ x^{-1} : x(U \cap V) \subset \mathbb{R}^n \to y(U \cap V) \subset \mathbb{R}^n$$

  is C^∞

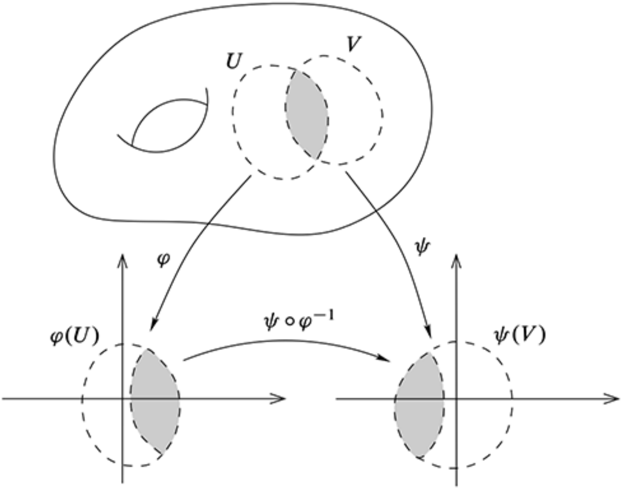

Figure: The transition map $\psi \circ \varphi^{-1}$ must have derivatives of all orders

Smooth Manifolds

A topological space (M, τ) together with a smooth atlas $\mathcal{A}$ is called a smooth manifold.
We also require that the topological space is Hausdorff for good properties to hold.

Examples of Manifolds

$M = \mathbb{R}^n$ with the standard topology and atlas given by just one chart $(U = \mathbb{R}^n, x = id)$ is trivially a smooth manifold.

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Every surface in R^3 is a manifold
Klein Bottle, torus, Mobius Strip ...

Functions And Curves

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We say that a curve $\gamma : I \subset \mathbb{R} \to M$ is of class C^∞ if $x \circ \gamma : I \subset \mathbb{R} \to x(M) \subset \mathbb{R}^n$ is of class C^∞ for all charts.

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Everything is well-defined if the atlas is smooth!!

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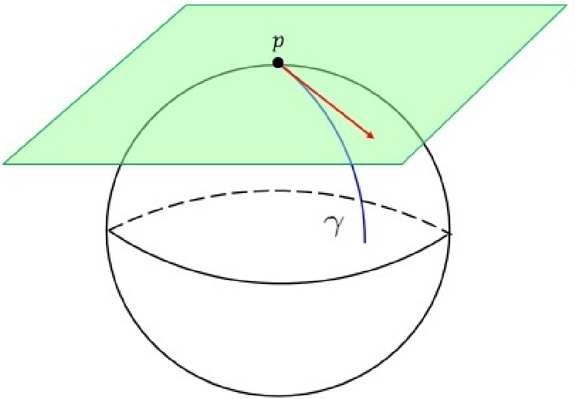
The transition map $y \circ x^{-1}$ is smooth, thus smoothness in γ is well-defined.

Tangent Vector



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The tangent vector to p, X_p, is the operator which maps smooth functions $f : M \to \mathbb{R}$ to number:

$$X_p : f \mapsto \frac{d(f \circ \gamma)}{dt}(t_0)$$

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Let's find a useful one:

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Remark: $x(p) = (x^1(p), ..., x^n(p))$

Some definitions

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The set of vectors $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\}$ are linearly independent!

Chart-induced basis

This definitions give us a nice expression:

$$X_p(f) = \frac{d(x \circ \gamma)^k}{dt} \frac{\partial (f \circ x^{-1})}{\partial x^k} = \dot{\gamma}^k \frac{\partial}{\partial x^k}(f)$$

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Thus $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\}$ is a basis for $T_p(M)$.

One-forms

One-forms

A one-form is a linear map $\omega : T_p(M) \to \mathbb{R}$, i.e, for all vectors $X, Y \in T_p(M)$ and $\alpha \in R$

$$\omega(X + \alpha Y) = \omega(X) + \alpha\omega(Y)$$

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With the point sum $(\omega^1 + \alpha\omega^2)(X) = \omega^1(X) + \alpha.\omega^2(X)$ the set of all one-forms $T_p^*(M)$ in $p \in M$ is a vector space.

Differential

The differential of function f is a one-form $df : T_p(M) \to \mathbb{R}$ such that

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The homeomorphism of a chart x gives rise to the differentials of the coordinate components dx^k

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The bases $\{\frac{\partial}{\partial x^k}\}$ and $\{dx^i\}$ are said to be dual:

$$dx^i \left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial x^k}(x^i) = \delta^i_j$$

Gradient of a Function

In particular, the differential of a function $f \in C^\infty(M)$ can be written as

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Thus we generalize the notion of gradient!

Tensors

Tensors

Let $p \in M$, a tensor of type (r, s) at p is a multilinear map

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Which means that T is linear in each argument:

For every vectors $X_k, X_l \in T_p$ and scalars $a, b \in \mathbb{R}$:

$$T(\omega^1, ..., \omega^r, X_1, ..., a.X_k + b.X_l, ..., X_s) =$$

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And the same for one-forms.

Tensor Operations

We can sum tensors and multiply by scalars:

$$(T + \alpha T')(\omega^1, ..., \omega^r, X_1, ..., X_s) =$$

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This give the structure of a vector space to the set $T^r_s(p)$ of all (r, s) tensors defined in $T^*_p \times ... \times T^*_p \times T_p \times ... \times T_p$

We can multiply a tensor by another tensor: If $R \in T^r_s$ and $S \in T^k_l$ are tensors, the tensor product $R \otimes S \in T^{r+k}_{s+l}$ is defined as

$$(R \otimes S)(\omega^1, ..., \omega^r, ...\omega^{r+k}, X_1, ..., X_s, ...X_{s+l}) =$$

$$R(\omega^1, ..., \omega^r, X_1, ..., X_s) \cdot S(\omega^{r+1}, ..., \omega^{r+k}, X_{s+1}, ..., X_{s+l})$$

Chart-induced basis, again...

If $\{\frac{\partial}{\partial x^i}\}$, $\{dx^i\}$ are dual basis of T_p and T_p^*, then the set:

$$\{\frac{\partial}{\partial x^{a_1}} \otimes ... \otimes \frac{\partial}{\partial x^{a_r}} \otimes dx^{b_1} \otimes ... \otimes dx^{b_s}\}$$

is a basis of $T_s^r(p)$, where every index $a_i, b_j \in \{1, ..., n\}$

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There will be n^{r+s} basis tensors

Thus every (r, s) tensor can be written as:

$$T = T^{a_1 \dots a_r}{}_{b_1 \dots b_s} \frac{\partial}{\partial x^{a_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{a_r}} \otimes dx^{b_1} \otimes \dots \otimes dx^{b_s}$$

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If it wasn't for Einstein's convention:

$$T = \sum_{a_1} \dots \sum_{a_r} \sum_{b_1} \dots \sum_{b_s} T^{a_1 \dots a_r}{}_{b_1 \dots b_s} \frac{\partial}{\partial x^{a_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{a_r}} \otimes dx^{b_1} \otimes \dots \otimes dx^{b_s}$$

Figure: Einstein dies of ligma

(skew) Symmetric part of a Tensor

Let T be a $(2,0)$ tensor, the symmetric part of $T^{a_1 a_2}$ is the (r,s) tensor defined as:

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The generalization for a (r,s) tensor is immediate

Tensor field

A vector field is an association $X : M \to \bigcup_p T_p(M)$ in a way that if $p \in M$ then $X(p) = X_p \in T_p(M)$

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A tensor field of type (r, s) is defined in a similar manner, where $T : M \ni p \mapsto T(p) \in T^r_s(p)$

Connection

Let X be a vector field and T a (r, s) tensor field defined in a smooth manifold M. A connection ∇ is a map:

$$\nabla : (X, T) \mapsto \nabla_X T$$

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It must satisfies the following properties:

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- If $f \in C^\infty(M)$, then $\nabla_X f = X(f)$

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For a vector field Y, we can calculate the covariant derivative as follows:

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$$X^i \frac{\partial}{\partial x^i}(Y^k) \frac{\partial}{\partial x^k} + X^i Y^k \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}$$

we don't know the precise form of $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}$, but we know that it's a vector (field)!

Connection coefficients

So we can expand $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}$ in a basis:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} = \Gamma^q{}_{ki} \frac{\partial}{\partial x^q}$$

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The $\Gamma^q{}_{ki}$ are called the connection coefficients.

They are the same for a one-form

With the same construction we can get, for one-forms ω:

$$\nabla_X \omega = \left(X^i \frac{\partial}{\partial x^i}(\omega_j) - X^i \omega_k {\Gamma^k}_{ji} \right) dx^j$$

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And for a general tensor field we apply the Leibniz rule many times using the same construction to get to the general formula:

$$(\nabla_X T)^{a_1 \dots a_r}{}_{b_1 \dots b_s} = X^i \frac{\partial}{\partial x^i} T^{a_1 \dots a_r}{}_{b_1 \dots b_s}$$

$$+ X^k T^{j \dots a_r}{}_{b_1 \dots b_s} \Gamma^{a_1}{}_{jk} + \text{all terms in the upper indices}$$

$$- X^i T^{a_1 \dots a_r}{}_{k \dots b_s} \Gamma^k{}_{b_1 i} - \text{all terms in the lower indices}$$

Parallel Transport

Let $\gamma : I \subset \mathbb{R} \to M$. The tangent vector field v_γ is such that if $p = \gamma(t)$ for some $t \in I$, then $v_\gamma(p) = \dot{\gamma}^k(p)\frac{\partial}{\partial x^k}$

Parallel Transport

Let $\gamma : I \subset \mathbb{R} \to M$ and v_γ be the tangent vector field defined as before. A vector field X is said to be parallelly transported along γ if

$$\nabla_{v_\gamma} X = 0$$

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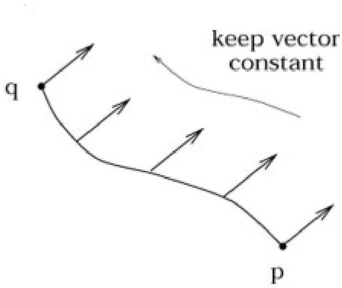

Autoparallel curves

The curve that moves as straight as possible has its tangent vector field parallelly transported along γ, therefore

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$$\ddot{\gamma}^q + \Gamma^q{}_{nm} \dot{\gamma}^m \dot{\gamma}^n = 0$$

Parallel transport

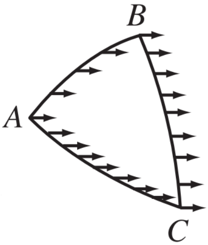


Figure: Parallel transport in flat space

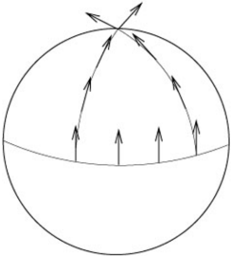
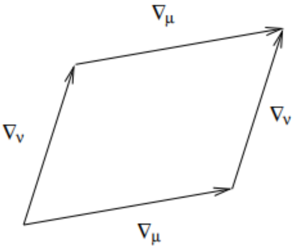


Figure: Parallel transport in $\mathcal{S}^2$

Curvature



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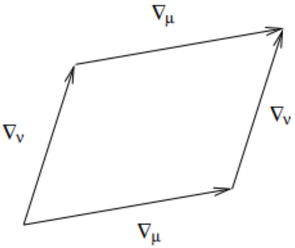

The non-commutative behavior of the covariant derivative measures the curvature:

$$[\nabla_{\frac{\partial}{\partial x^\mu}}, \nabla_{\frac{\partial}{\partial x^\nu}}] \neq 0$$

Let V be a vector field, then

$$\left([\nabla_{\frac{\partial}{\partial x^\mu}}, \nabla_{\frac{\partial}{\partial x^\nu}}]V\right)^\rho =$$

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$$\left(\frac{\partial}{\partial x^\mu}\Gamma^\rho{}_{\nu\sigma} - \frac{\partial}{\partial x^\nu}\Gamma_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}\right) V^\sigma - 2\Gamma^\lambda{}_{[\mu\nu]}(\nabla_{\frac{\partial}{\partial x^\lambda}} V)$$

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If $\Gamma^\lambda{}_{[\mu\nu]} = 0$ the curvature is contained in the proportional term , the Riemann tensor:

$$R^\rho{}_{\sigma\mu\nu} = \frac{\partial}{\partial x^\mu}\Gamma^\rho{}_{\nu\sigma} - \frac{\partial}{\partial x^\nu}\Gamma_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}$$

Torsion tensor and Riemann tensor

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$$T^a{}_{bc} = 2.\Gamma^a{}_{[bc]} = 0 \implies \Gamma^a{}_{bc} = \Gamma^a{}_{cb}$$

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The Riemann tensor is the $(1,3)$ tensor field

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In components of a chart:

$$R^a{}_{bcd} = \frac{\partial}{\partial x^c}(\Gamma^a{}_{db}) - \frac{\partial}{\partial x^d}(\Gamma^a{}_{cb}) + \Gamma^a{}_{cf}\Gamma^f{}_{db} - \Gamma^a{}_{df}\Gamma^f{}_{cb}$$

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The contraction of the Riemann tensor gives the Ricci tensor:

$$R_{ab} = R^c{}_{acb}$$

Useful Identities

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Later we'll see the geometrical significance of the Riemann tensor

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For every point we associate smoothly a symmetric non-degenerate bilinear form in $T_p(M)$.

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Therefore we have an 'isomorphism' between the space of all vector fields and the space of all one-form fields: $\omega_a = g_{ab} X^b$

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A smooth n-dimensional manifold with a metric is a pseudo Riemannian manifold if $s < n$

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A metric on a n-dimensional smooth manifold is a Lorentz metric if $s = n - 2$

$$g_{ij} = diag(+1, ..., +1, -1)$$

Connection from the metric

With a metric we can define a unique torsion-free connection with the compatibility condition:

$$\nabla_X g = 0 \text{ for all vector fields } X$$

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We can show that the connection coefficients satisfies:

$$\Gamma^q{}_{ij} = \frac{1}{2} g^{qm} \left(\frac{\partial}{\partial x^i} g_{mj} + \frac{\partial}{\partial x^j} g_{mi} - \frac{\partial}{\partial x^m} g_{ij} \right)$$

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Facts about the Riemann Tensor

- If the Riemann tensor vanishes in a simply-connected region, we can construct a chart (U, x) where the g_{ij} are constants in U
- Considering the index symmetries in the Riemann tensor, there exist $\frac{1}{12}n^2(n^2-1)$ independent components
- The scalar curvature is defined as $R = g^{ab}R_{ab}$

Geodesics

If $\gamma : \mathbb{R} \to M$ is a curve on a smooth manifold with a metric g, the length of the path between two points $\gamma(t_0) = p$, $\gamma(t) = q$ is:

$$L = \int_{t_0}^{t} \sqrt{|g(v_\gamma, v_\gamma)|} dt$$

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A geodesic is a stationary curve of the L functional.

Some thoughts on Minkowski Spacetime

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Minkowski spacetime is a four dimensional smooth manifold M with a Lorentz metric η such that everywhere:

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i.e, it takes the diagonal form

$$\eta_{ij} = diag(-1, 1, 1, 1)$$

in a unique chart that covers M.

Causal structure

For every vector field X in spacetime we have:

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For every vector $X \in T_p(M)$

- If $\eta(X, X) > 0$, X is said to be spacelike
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The proper time is *represented* by

$$d\tau^2 = -\eta_{ij} dx^i dx^j$$

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so does the riemannian curvature tensor $R^{a}{}_{bcd} = 0$

Geodesics

In a chart (U, x) that cover the path, the geodesic equation satisfies:

$$\delta L = \int \frac{1}{2\sqrt{g(v_\gamma, v_\gamma)}} \delta g(v_\gamma, v_\gamma) dt = 0$$

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The problem reduces to $\mathcal{L} = \frac{1}{2} g(v_\gamma, v_\gamma) = \frac{1}{2} g_{ij} \dot{\gamma}^i \dot{\gamma}^j$

Geodesics

Using the Euler-Lagrange equations...

$$\frac{d^2\gamma^q}{d\tau^2} + \frac{1}{2}g^{qm}(\frac{\partial}{\partial x^i}g_{mj} + \frac{\partial}{\partial x^j}g_{mi} - \frac{\partial}{\partial x^m}g_{ij})\frac{d\gamma^j}{d\tau}\frac{d\gamma^k}{d\tau} = 0$$

Aha!

$$\frac{d^2\gamma^q}{d\tau^2} + \Gamma^q{}_{ij}\frac{d\gamma^j}{d\tau}\frac{d\gamma^k}{d\tau} = 0$$

A null geodesic cannot be parametrized by the proper time.

Spacetime

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The relation of the curvature with the energy-momentum tensor T_{ab} is given by the Einstein's Field Equations

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$$

That's all folks !