Topology (and Metrics) for the Young at Heart

Nicholas Funari Voltani

March 21, 2018

1 A motivation

While working with the real numbers $\mathbb{R}$, one eventually finds the concept of the absolute value $|.|$. But, rather unexpectedly (for the beginner, at least), this function which spits out the positive value of a number can also be perceived as some sort of *distance* between numbers. As time went on, this idea was naturally transplanted to higher-dimensional spaces $\mathbb{R}^n$ (as well as $\mathbb{C}^n$) with Pythagoras' Theorem.

Of the characteristics found in these "distance" functions, one finds that they're always greater than, or equal to, 0 (one doesn't talk about negative distances, no one walks -4 meters in a casual stroll); also, they satisfy the so-called *triangle inequality*, in which the distance from two points is always less than, or equal to, the sum of the distances to a third point (as seen with the absolute-value function). Also some obvious facts, such that "the distance from a point to itself is 0" and "points have a 0 distance between each other if they're the same point", and "distance from a to b is the same as from b to a".

Also in $\mathbb{R}$, one finds the concept of open sets, which makes use of the order relation $a < b$ in $\mathbb{R}$ and the absolute-value distances to talk about this "room without edges" inbetween a and b. This concept was also transported to higher-dimensional analogs (called "open balls"), using its concept of distance inherited by $\mathbb{R}$.

One finds that, while working with open sets in $\mathbb{R}^n$, some peculiar properties arise: while the arbitrary union of open sets remains open, only *finitely* many open sets can be intersected with each other in order to remain open (think, for instance, on the countable intersection of open sets $(0, \frac{1}{n})$).

What makes the absolute value function a "distance" function? What if we're in a set in which there isn't any "signs"? What makes this "room without edges" an open set? What about losangles without edges, are they open? What if there isn't a distance defined in this space, can there not be any open sets of sorts? One can clearly see an opportunity to generalize these concepts so characteristic of $\mathbb{R}$ to different spaces, and then recovering them as particular cases of a more general rule.

2 Ready to Start: Metric Spaces

Definition 2.1 (Metric Spaces)**.** *Given a set $M \neq \emptyset$, a function $d : M \times M \to \mathbb{R}_+$ is called a metric if it satisfies the following properties:*

1. *$\forall x, y \in M$, $d(x, y) = 0 \iff x = y$ (so-called "non-degeneracy");*

2. *$\forall x, y, z \in M$, $d(x, z) \leq d(x, y) + d(y, z)$ ("triangle inequality");*

3. *$d(x, y) \geq 0$ (so-called "positivity");*

4. *d(x,y) = d(y,x) ("commutativity");*

The pair (M, d) is called a metric space if d is a metric.

One can also require only the first two conditions (non-degeneracy and the triangle inequality), and retrieve the last two as results. The four axioms are mostly historical, and alluding to the characteristics found in $\mathbb{R}$.

Given this general definition, one can then affirm that $(\mathbb{R}, |.|)$ is a metric space, as well as $(\mathbb{R}^n, d_E)$ (where d_E is the Euclidean distance).

Now, one can construct new spaces that don't look any similar to $\mathbb{R}$, and yet talk about distances the same way as $\mathbb{R}$. For instance, consider the following metric space:

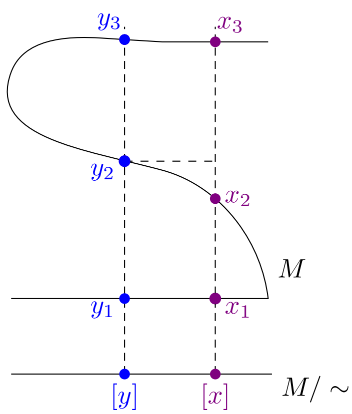
Given any $M \neq \emptyset$, define d_t as $d_t(x, y) = \begin{cases} 0, \text{if } x = y; \\ 1, \text{if } x \neq y \end{cases}$ (it's often called the "trivial metric").

That is a metric, given the definition 2.1 (do give it a check!), but it feels...wrong, right? That's the price of generalizations, although they're not impeditive to our work in any way; just interesting manifestations of a fortuitous definition.

2.1 Pseudometrics

One can even loosen up the condition of non-degeneracy, and create the so-called "pseudometrics": different points can have null distance. An example is due:

Exemple 1 (A snake-y boi[1])**.** Consider the following figure, with the space $M \subset \mathbb{R}^2$ as shown.



Define a "distance" between points as the distance between their shadows (light coming from above). Intuitively, it satisfies the triangle inequality (it's just the usual metric in $\mathbb{R}$), as well as positivity and commutativity, but note that points above each other have distance 0 while being distinct points. Thus, it is a pseudometric (denote it $\tilde{d}$).

What one can do, if it's an annoyance, is to identify (via an equivalence relation $\sim$) points that have distance 0 with the same "tag" (equivalence relation $[.]$). The quotient space $M/\sim$ is just a line, as the points above each other have been "compactified" by the identification. Note that one can define a metric in this new "compactified" space, by the following procedure:

Given $x_1, x_2 \in [x]$, $y_1, y_2 \in [y]$ (so $\tilde{d}(x_1, x_2) = \tilde{d}(y_1, y_2) = 0$), define $d([x], [y]) := \tilde{d}(x_1, y_1)$, which is independent of the representatives of the tags, because

$$\begin{aligned} \tilde{d}(x_1, y_1) &\leq \tilde{d}(x_1, x_2) + \tilde{d}(x_2, y_2) + \tilde{d}(y_2, y_1) = \tilde{d}(x_2, y_2) \\ \tilde{d}(x_2, y_2) &\leq \tilde{d}(x_2, x_1) + \tilde{d}(x_1, y_1) + \tilde{d}(y_1, y_2) = \tilde{d}(x_1, y_1) \\ &\therefore \tilde{d}(x_1, y_1) = \tilde{d}(x_2, y_2) \end{aligned}$$

Remark. Note that, in our example, this is the same as saying that the pseudodistances in the last statement are the same due to the fact that going from $x_1[y_1]$ to $x_2[y_2]$ keeps its "shadow" in the same place.

This is a general result, applying to any pseudometric space $(M, \tilde{d})$, and producing a metric space $(M/\sim, d)$ as done above.

2.2 Open Balls, Open Sets and Continuity in Metric Spaces

Given a metric space (M, d), one can define the open balls to be $B_\delta(x) := \{y \in M \mid d(x, y) < \delta\}$. It can be checked that arbitrary unions of these open balls remain open, and that at most finitely many open balls can intersect into an open ball (for the countable counterexample, one can simply think about a ball with shrinking radius of $\frac{1}{n}, \forall n \in \mathbb{N}$).[2]

[1] Special thanks to Felipe Dilho.

[2] This requires that singleton sets $\{x\}$ are closed in metric spaces; but that's clear, since one cannot construct an open ball entirely inside $\{x\}$. Another way to look at it is to think about its complement $X \setminus \{x\}$, whose open balls around the points $y \in X \setminus \{x\}$ can lie entirely within it with a radius $\frac{d(x,y)}{2}$, for instance.

Inspired by the definition of continuity in $\mathbb{R}$ (like everything done so far), one can think of maps between metric spaces $f : M_1 \to M_2$ to be continuous if, for every $\epsilon > 0$, there is some $\delta > 0$ such that $0 < d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \epsilon$. An equivalent definition is to require that $\forall B \in \tau_2, f^{-1}(B) \in \tau_1$. [3]

It is also worth noting that a same set can have different metrics and produce different metric spaces. For instance, a sequence may converge in $M \neq \emptyset$ with respect to some metric d_1, but not for d_2, even though these metric are defined on the same set M. [4] However, if the metrics are equivalent under the equivalence relation

$$d_1 \sim d_2 \iff \exists \alpha_1, \alpha_2 \geq 0 \mid \forall x, y \in M, \alpha_1 d_1(x, y) \leq d_2(x, y) \leq \alpha_2 d_1(x, y)$$

then sequences converging in (M, d_1) will converge in (M, d_2), vice-versa (showing this is exercise **1**).

3 Topological Spaces

Definition 3.1 (Topological Spaces)**.** *Given a set $X \neq \emptyset$, $\tau \subset \mathcal{P}(X)$ a collection of subsets of X is called a topology if*

1. $\emptyset, X \in \tau$;
2. $\forall \{A_\lambda\}_{\lambda \in \Lambda} \subset \tau$, $\bigcup_{\lambda \in \Lambda} A_\lambda \in \tau$ *(τ is closed under arbitrary unions);*
3. $\forall \{A_k\}_{k=1}^{n} \subset \tau$, $\bigcap_{k=1}^{n} A_k \in \tau$ *(τ is closed under finite intersections).*

The pair (X, τ) is called a topological space if τ is a topology. Sets inside τ are often called "open sets".

Given this definition, note that every metric space is automatically a topological space, because of its induced topology of open balls; that is the case of $\mathbb{R}^n$.

Given this definition, one can create topological spaces where its open sets aren't what we're usually accostumed to. For instance, given $M \neq \emptyset$, consider $\tau_c := \{\emptyset, M\}$; it's called the chaotic topology, and it's the coarsest topology on M; there is also $\tau_d := \mathcal{P}(X)$, which is X's powerset, and in this case called the discrete topology, the largest topology on M. Do take notice that, then, $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ is a topological space, and in it even "closed" sets such as $[0, 1]$ are elements of the topology, and thus called "open"! When talking about open sets, do take much care with respect to which topology it belongs, although usually it'll be implicit, given the context.

We have the definition of open sets; what about closed ones?

Definition 3.2 (Closed sets)**.** *Given (X, τ) topological space, a so-called "closed set" $F \subset X$ is such that $X \setminus F \in \tau$ (its complement is open). One can define the collection of closed sets $\mathcal{F}(\tau) := \{F \subset X \mid X \setminus F \in \tau\} \subset \mathcal{P}(X)$, which trivially satisfies*

1. $\emptyset$, $X \in \mathcal{F}(\tau)$;
2. $\forall \{F_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{F}(\tau)$, $\bigcap_{\lambda \in \Lambda} F_\lambda \in \mathcal{F}(\tau)$ *(τ is closed under arbitrary intersections);*
3. $\forall \{F_k\}_{k=1}^{n} \subset \mathcal{F}(\tau)$, $\bigcup_{k=1}^{n} F_k \in \mathcal{F}(\tau)$ *(τ is closed under finite unions).*

An important idea in topology is that of a *topological subspace*, which is a way to construct smaller spaces, given a starting topological space.

Definition 3.3 (Topological Subspace)**.** *Let (X, τ) be a topological space, and $Y \subset X$. Then (Y, τ_Y) is called a topological subspace of X, where $\tau_Y = \{A \cap Y\}_{A \in \tau}$ is the topology induced by Y (one can check that it's indeed a topology). It's basically a collection of the "shadows" of the open sets in τ "flying over" Y.*

[3] It's worth noting that some of these open balls B may have an empty preimage, if f doesn't reach it.

[4] A sequence $(x_n)_{n \in \mathbb{N}} \subset M$ converges to a point $x \in M$ if, for every $\epsilon > 0$, $\exists N \in \mathbb{N} \mid \forall n \geq N, d(x_n, x) < \epsilon$.

A very important concept in topological spaces is that of the interior and closure of subsets.

Definition 3.4 (Interior and Closure)**.** *Given a topological space* (X, τ)*, and* $Z \subset X$*, one can define its interior as*

$$\mathring{Z} := \bigcup_{\substack{A \in \tau \\ A \subset Z}} A$$

and its closure as

$$\overline{Z} := \bigcap_{\substack{F \in \mathcal{F}(\tau) \\ Z \subset F}} F$$

Remark. *Essentially, the interior of* Z *consists on "inflating"* Z *with open sets from the inside, while its closure consists on "crunching"* Z *from the outside with closed sets. Also, some authors denote the interior of* Z *by* $Int(Z)$ *and its closure by* $Cl(Z)$*.*

One can also define continuity of functions between topological spaces: a map between topological spaces $f : X_1 \to X_2$ is continuous if $\forall U \in \tau_2, f^{-1}(U) \in \tau_1$. Note that this clearly takes inspiration in the result from metric spaces, and makes no mention of distances, ϵ and δ. We abdicate the right to talk about distances between individual points, and now refer to *neighborhoods* around points.

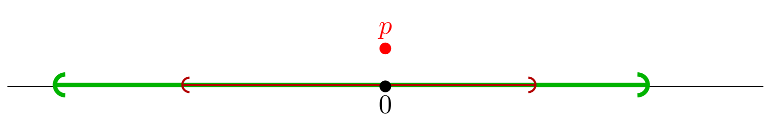
While on the subject of continuity, there is a way to define equivalence between topological sets (X_1, τ_1), (X_2, τ_2), by requiring a continuous bijection (with continuous inverse) between them (it's an easy check that it is indeed an equivalence relation); a function that satisfies this is called a *homeomorphism* between these spaces. There are some topological properties which are preserved via homeomorphisms (as in X_1 has P if and only if X_2 has P), as shall be seen below.

4 All of [not really] the Angels: Topological Properties

4.1 Hausdorff property of separation

Given a metric space (M, d), we also have a topology associated to d. Note also that, given $x, y \in M$, one can easily produce disjoint balls of radius $\frac{d(x,y)}{2}$ around x and y. This satisfies the Hausdorff topological condition, that different points in the space can be "separated" by disjoint open sets. So we can then correct our previous statement: a metric space is automatically a *Hausdorff* topological space. This is a property we hardly give any credit for, but which can be absent in arbitrary topological spaces. An example follows:

Exemple 2 (Double-origin $\mathbb{R}$ line)**.** Let $X = \mathbb{R} \cup \{p\}$ (p is just a non-$\mathbb{R}$ object, need not be a number even), and define its topology to be generated by the usual open sets in $\mathbb{R}$, but for every open set (a, b) which contains the origin 0, we create another open set $(a, 0) \cup \{p\} \cup (0, b)$. This does compose a topology, and it is *not* Hausdorff, since for $\{p\} \in (a_p, 0) \cup \{p\} \cup (0, b_p)$ and $0 \in (a_0, b_0)$, there will inevitably be some overlap. Thus, 0 and $\{p\}$ cannot be "separated" in the Hausdorff sense.



4.2 Connectedness & Path-Connectedness

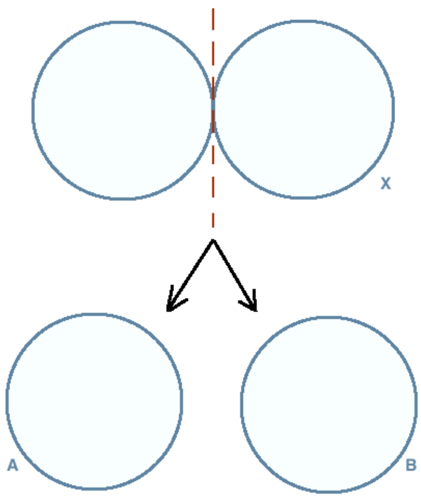
Given a topological space (X, τ), one usually defines when it's *disconnected*:

Definition 4.1 (Connected Spaces, usual definition)**.** (X, τ) *is said to be disconnected if* $\exists A, B \in \tau$ *such that* $X = A \cup B$*, where* $A, B \neq \emptyset, X$ *(*X *is composed of smaller, non-empty open sets). Then, one says that* X *is connected if it isn't disconnected.*

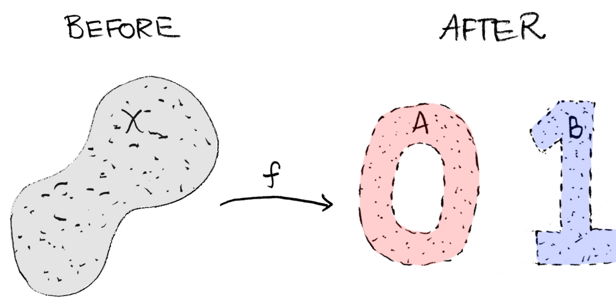
However, there is an alternative, equivalent definition:

Definition 4.2 (Connected Spaces, alternative definition)**.** *(X, τ) is said to be disconnected if there is a continuous surjection $f : X \to \{0, 1\}$, where $\{0, 1\}$ is given its discrete topology $\tau_{\{0,1\}} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. On the other hand, X is connected if it's not disconnected, i.e., if there is no way to construct a continuous surjection between (X, τ) and $(\{0, 1\}, \tau_{\{0,1\}})$; that is to say, every continuous map $f : X \to \{0, 1\}$ need be a constant map.*

Remark. This definition at first may appear overwhelming, but it's the most natural, since the function tells whether X can have "distinguishable" points (disconnected) or not (connected).



(a) A disconnected space, according to definition 4.1.



(b) A disconnected space, according to definition 4.2; the shapes of 0 and 1 are obviously just a reference to f's image points.

This is a topological property, i.e., it's preserved by homeomorphisms, as proven in the following

Theorem 4.1. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function. If X is connected, then $f(X)$ will also be connected.*

Proof. This is equivalent to proving that $f(X)$ is disconnected $\implies$ X is disconnected (due to the contrapositive). So suppose at first that $f(X)$ is disconnected.

$$f(X) = A \cup B \implies X = (f^{-1})^{-1}(X) \subset f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(f(X)) \subset X$$
$$\therefore X = f^{-1}(A) \cup f^{-1}(B)$$

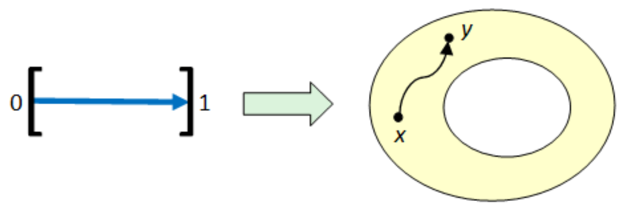
and thus, since f is continuous and open sets' preimages are also open, X is disconnected. □

In Calculus, however, one generally works with the idea of *path-connectedness*:

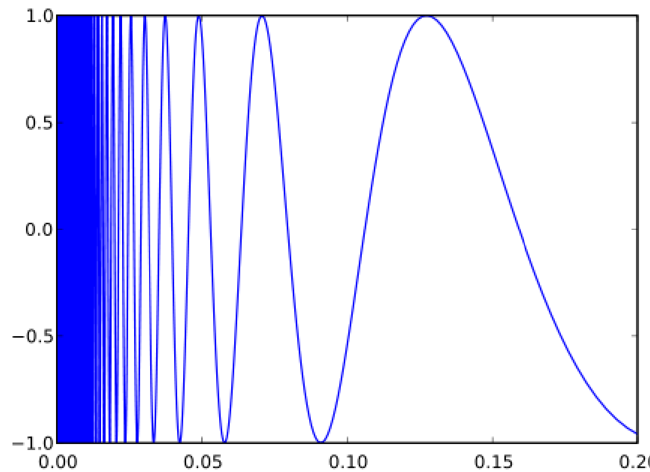
Definition 4.3 (Path-connectedness)**.** *(X, τ) is said to be path-connected if $\forall x, y \in X$, there is a continuous path $\gamma : [0, 1] \to X, \gamma(0) = x, \gamma(1) = y$ connecting x and y.*

This idea is stronger than that of a connected space, since a disconnected space can't have a continuous path connecting its disjoint "sections". However, there are spaces which are connected, but **not** path-connected.

Exemple 3 (Topologist's Sine Curve)**.** Consider the graph of the function $(x, \sin(\frac{1}{x}))$ in the interval $(0, 1]$, united with, say, $(0, 0)$, as in fig. 2b. Note that, if we attempt to surround $(0, 0)$ with an open set (we're assuming this space is inside $\mathbb{R}^2$), it'll inevitably contain some points from the graph, so it must be connected. However, there is no continuous path that takes a point $(x, \sin(\frac{1}{x}))$ to $(0, 0)$ in a finite parameter range like $[0, 1]$, since $\sin(\frac{1}{x})$ isn't defined in $x = 0$. It's like it "takes forever" for $(x, \sin(\frac{1}{x}))$ to reach the y-axis, and so this space isn't path-connected.



(a) A path-connected space.



(b) The topologist's sine curve: the graph $(x, \sin(\frac{1}{x}))$, $x \in (0, 1]$, union with the point $\{0.0\}$ (but just that point on the y-axis!).

4.3 Compacticity

Definition 4.4 (Compacticity)**.** *Given a topological space (X, τ), a subset $K \subset X$ is said to be compact if for every open cover of K, there is a finite sub-cover of K. That is to say,*

$$\forall \mathcal{A} = \{A_\alpha\}_{\alpha\in\Gamma} \subset \tau \mid K \subset \bigcup_{\alpha\in\Gamma} A_\alpha$$

$$\implies \exists \tilde{\mathcal{A}} = \{A_k\}_{k=1}^{n} \subset \mathcal{A} \mid K \subset \bigcup_{k=1}^{n} A_k$$

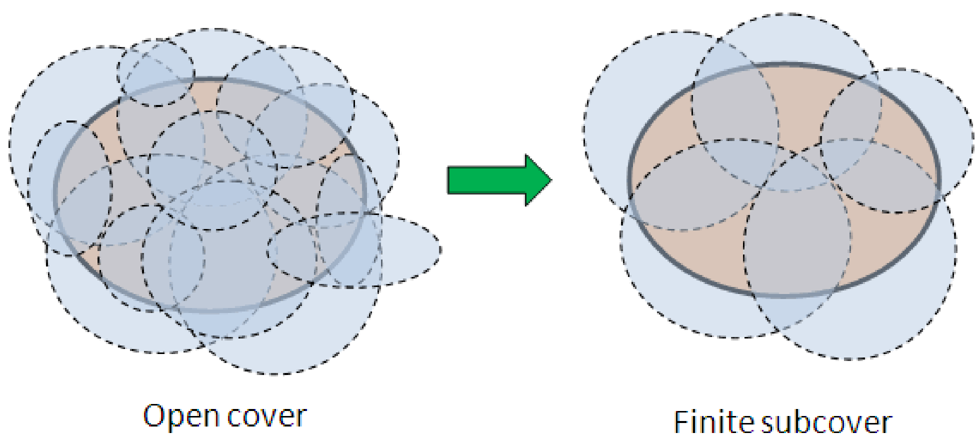


Figure 3: If K is compact, then *any* open cover (even an uncountable one!) will have a finite open subcover; this image shows only *one* such open cover of K (and it's already finite, so it's just for show).

The following analogy may be of use: if a city is compact with above's definition, then no matter how many policemen are hired (even an uncountable amount of cops; that's a lotta cops!), there is a *finite* amount of them which will be enough to keep track of the city's whole area. [5]

[5]This is some mathematician's quote, but I couldn't find who...I just love it too much, and had to share it.

But do note that it's still required to give some amount of open sets that cover your compact set, and *then* one can affirm that, out of those open sets, there is a finite amount which will still do the job of covering the whole thing.

Compacticity is also a topological property, because of the following

Theorem 4.2. *Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function. If $K \subset X$ is compact, then $f(K)$ will also be compact.*

Proof. Let $\mathcal{A} = \{A_\lambda\}_{\lambda\in\Lambda} \subset \tau_Y$ be an open cover of $f(K)$. Then $f(K) \subset \bigcup_{\lambda\in\Lambda} A_\lambda \implies K \subset \bigcup_{\lambda\in\Lambda} f^{-1}(A_\lambda)$. K is compact, so there is a finite subcover $\tilde{\mathcal{A}} = \{A_k\}_{k=1}^n \subset \mathcal{A} \mid K \subset \bigcup_{k=1}^{n} f^{-1}(A_k)$. Thus, $f(K) \subset \bigcup_{k=1}^{n} A_k$, and $f(K)$ is compact. □

In Calculus, one often learns that "compact sets are closed and bounded", like $[a, b]$. However, it's true because of the Heine-Borel Theorem, which states that in $\mathbb{R}^n$, compact sets are closed and bounded, and vice-versa. Remember that in general topological spaces, one doesn't necessarily have the idea of a "bounded" set.

5 The Working Hour

1. Show that, given a set $M \neq \emptyset$, and equivalent metrics $d_1 \sim d_2$, then

1. $\tau_1 = \tau_2$ (they induce the same topology);
2. Given $(x_n)_{n\in\mathbb{N}} \subset M$ sequence, $x_n \overset{d_1}{\to} x \iff x_n \overset{d_2}{\to} x$ (sequences converge to same point in these spaces).

(*Hint*: Maybe it'd be useful to show that $\sim$ is indeed an equivalence relation, and that $d_1 \sim d_2 \iff d_2 \sim d_1$.)
2. Show that, given a set $X \neq \emptyset$, and topologies τ_1, τ_2 in X, then $\tau_1 \subset \tau_2 \iff \mathcal{F}(\tau_1) \subset \mathcal{F}(\tau_2)$.
3. Show that, given a set $X \neq \emptyset$, $Z \subset X$, and topologies τ_1, τ_2 in X such that $\tau_1 \subset \tau_2$, then $Int_1(Z) \subset Int_2(Z)$, and $Cl_2(Z) \subset Cl_1(Z)$. [6]
4. All functions $f : (X, \mathcal{P}(X)) \to (Y, \tau_Y)$ (where $\mathcal{P}(X)$ is X's power set/discrete topology, and τ_Y some topology in Y) are continuous.
5. Show that, given continuous functions $f : (X, \tau_X) \to (Y, \tau_Y)$, $g : (Y, \tau_Y) \to (Z, \tau_Z)$, their composition $g \circ f : (X, \tau_X) \to (Z, \tau_Z)$ is also continuous.
6. Show that, if (X_1, τ_1), (X_2, τ_2) are equivalent topological spaces, then images of open sets are also open.
7. Show that definitions 4.1 and 4.2 are indeed equivalent.
8. Prove that every space X with the chaotic topology $\tau_{ch} = \{\emptyset, X\}$ is connected, and every space with the discrete topology $X = \mathcal{P}(X)$ is disconnected.
9. Prove that a space (X, τ) is connected if, and only if, only X and $\emptyset$ are open and closed in τ. [7]
10. Show that, given a topological space (X, τ), if $x, y \in X$ are connected by a continuous path $\gamma_1 : [0, 1] \to X$, and $y, z \in X$ are also connected by a continuous path $\gamma_2 : [0, 1] \to X$, then there exists a continuous path connecting x and z. Construct it explicitly.
11. Show that the union of a family of connected sets $\{A_i\}_{i\in\mathcal{I}}$ is connected, if $A_i \cap A_j \neq \emptyset, \forall i, j \in \mathcal{I}$. (It may be easier to work with definition 4.2.)
12. Show that, if A is connected, then $\overline{A}$ is also connected. (Remember about connectedness in subspaces! Maybe use the contrapositive...?)
13. Show that the *finite* union of compact sets is also compact. Show an example in $\mathbb{R}$ where a countable union of compact sets fails to be compact.

[6] Note that, by using larger topologies, the interior and the closure of a set "tighten" more closely to Z, until $\tau = \mathcal{P}(X)$, with which $Int(Z) = Cl(Z) = Z$, and then all open sets are also closed sets.

While using coarser topologies, the "format" of Z's interior and closure get less intricate, until they're trivially $Z = \overline{Z}$, $\mathring{Z} = \emptyset$ (for $Z \subset X$) in $\tau = \{\emptyset, X\}$, in which only X is open, and all its subsets (including itself) are closed.

[7] Some are depraved enough to call them *clopen*. Yikes.

6 Further Readings

- *Introduction to Metric and Topological Spaces*, Wilson A. Sutherland. Oxford University Press, 1975.
- *A Few Of My Favorite Spaces: The Topologist's Sine Curve*, Evelyn Lamb https://blogs.scientificamerican.com/roots-of-unity/a-few-of-my-favorite-spaces-the-topologist-s-sine-curve/
- *A Few of My Favorite Spaces: The Line with 2 Origins*, Evelyn Lamb https://blogs.scientificamerican.com/roots-of-unity/a-few-of-my-favorite-spaces-the-line-with-2-origins/
- *Notas para um Curso de Física-Matemática*, João Carlos Alves Barata.
  - Cap. 27: "Espaços Métricos" http://denebola.if.usp.br/ jbarata/Notas$_d e_a ula/arquivos/nc - cap27.pdf$
  - Cap. 29: "Espaços Topológicos e Espaços Mensuráveis. Definições e Propriedades Básicas" http://denebola.if.usp.br/ jbarata/Notas$_d e_a ula/arquivos/nc - cap29.pdf$
  - Cap. 32: "Continuidade e Convergência em Espaços Topológicos" http://denebola.if.usp.br/ jbarata/Notas$_d e_a ula/arquivos/nc - cap32.pdf$
  - Cap. 34: "Alguns Tópicos Especiais em Topologia e Análise" http://denebola.if.usp.br/ jbarata/Notas$_d e_a ula/arquivos/nc - cap34.pdf$