Topology (and Metrics) for the Young at Heart

Nicholas Funari Voltani

March 21, 2019

Motivation

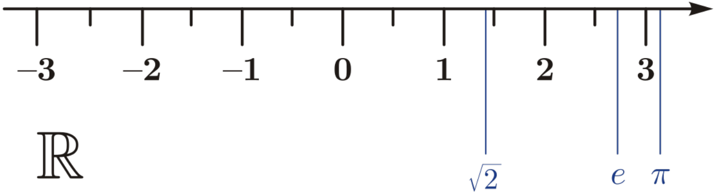
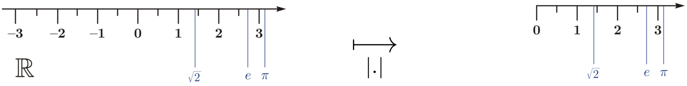


Figure: Our beloved $\mathbb{R}$ line.

Absolute Value



$x \in \mathbb{R}$

$|x| \in \mathbb{R}_+$

Absolute Value & Distances

$|x - y|$: distance between $x, y \in \mathbb{R}$.

Absolute Value & Distances

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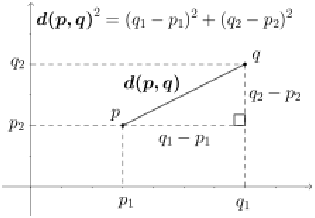
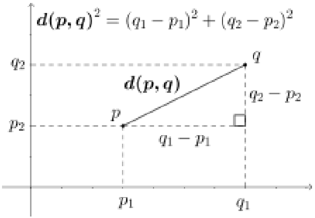
- $|x - y| \geq 0$ (Positive distances);
- $|x - y| = |y - x|$ (Commutativity)

Distances in $\mathbb{R}^n$



Euclidean distance in $\mathbb{R}^2$

Distances in $\mathbb{R}^n$

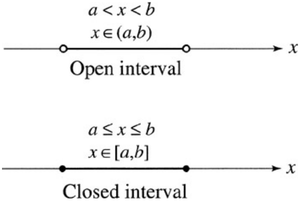


Euclidean distance in $\mathbb{R}^2$

Same idea for $\mathbb{R}^n$! (Just the size of the n-rectangle's diagonal)

Open balls in $\mathbb{R}^n$

In $\mathbb{R}$:



In $\mathbb{R}^n$:

$$B_\delta(\vec{x}) := \{\vec{y} \in \mathbb{R} \mid ||\vec{x} - \vec{y}||_E < \delta\}$$

Open sets in $\mathbb{R}^n$

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$$\forall x \in A, \exists \delta_x > 0 \mid B_{\delta_x}(x) \subset A.$$

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- Arbitrary unions of open sets is an open set;
- Finite intersections of open sets is an open set;
    - Even a countable intersection may fail to be open: $\bigcap_{n \in \mathbb{N}} (-1, \frac{1}{n}) = (-1, 0]$

Definition: Metric Spaces

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The pair (M, d) is called a *metric space* when d is a metric.

Some Classic Metric Spaces

- $(\mathbb{R}, |.|)$ and $(\mathbb{R}^n, d_E)$, $d_E(\vec{x}, \vec{y}) = \sqrt{\sum_{k=1}^{n} (x_i - y_i)^2}$

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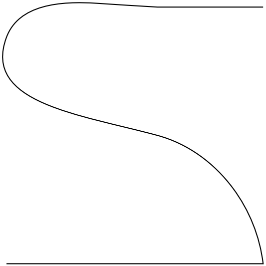
- $(\mathbb{R}^n, d_1)$, where $d_1(\vec{x}, \vec{y}) := \sum_{i=1}^{n} |x_i - y_i|$
- $(\mathcal{C}([a,b]), d_\infty)$, where $d_\infty(f,g) := \sup_{x \in [a,b]} |f(x) - g(x)|$

Pseudometrics: A case



Consider this snake-y boi, M.

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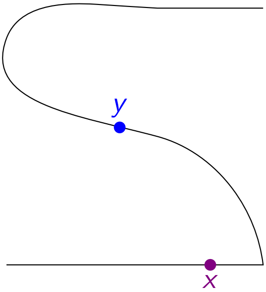

Let $x, y \in M$.

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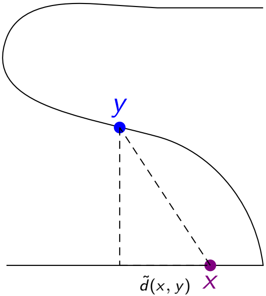
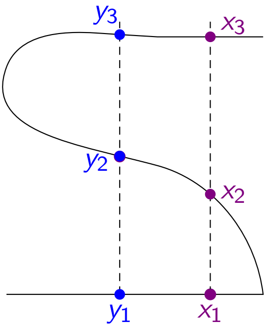


Consider $\tilde{d}(x, y)$ to be the horizontal “distance” between x and y.

Pseudometrics: A case

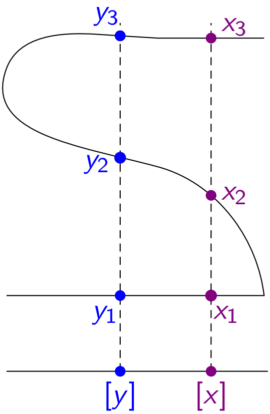


Distinct points above each other have "distance" 0!

$\therefore \tilde{d}$ isn't a metric!

But it satisfies the other properties...

Creating Metrics from Pseudometrics

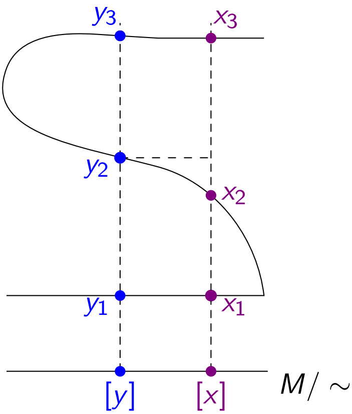


Let's identify points:

$$a \sim b \iff \tilde{d}(a, b) = 0$$

$$[x] = \{z \in M \mid \tilde{d}(x, z) = 0\}$$
$$= \{z \in M \mid z \sim x\}$$

Creating Metrics from Pseudometrics



Define $d([x],[y]) := \tilde{d}(x,y)$. It is well-defined, since

$$\tilde{d}(x_1,y_1) \leq \tilde{d}(x_2,y_2)$$
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$$\therefore \tilde{d}(x_1,y_1) = \tilde{d}(x_2,y_2)$$

$\therefore (M/\sim, d)$ is a metric space.

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$f : (M, d_M) \to (N, d_N)$ is continuous in $x \in M$ if
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There are naturally open balls of the form
$B_\delta(x) = \{y \in M \mid d(x, y) < \delta\}$.
Definitions of open and closed sets is similar to that of $\mathbb{R}^n$.

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Note that *"open-ness"/"closed-ness"* is always with respect to some topology (also obviously w.r.t the space X).

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- $(\mathbb{R}^n, \tau_{std})$ (surprise, surprise);
- Sierpinski space $\mathbb{S} = \{0, 1\}, \tau_{\mathbb{S}} = \{\emptyset, \{1\}, \{0, 1\}\}$ is a topological space!
- Metric spaces (M, d), with their induced topology by the metric τ_d can be seen as topological spaces.

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Continuity in Topological Spaces

A map $f : (X, \tau_X) \to (Y, \tau_Y)$ is said to be continuous if, for every $V \in \tau_Y$, $f^{-1}(V) \in \tau_X$.
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A bijection $X \overset{\varphi}{\longleftrightarrow} Y$ is said to be a *homeomorphism* if it's continuous both ways.
We say $X \sim Y$ if there's a homeomorphism between them; they're said to be *topologically equivalent*.

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Homeomorphisms are important objects in Topology, since they preserve many topological properties.

The Hausdorff property

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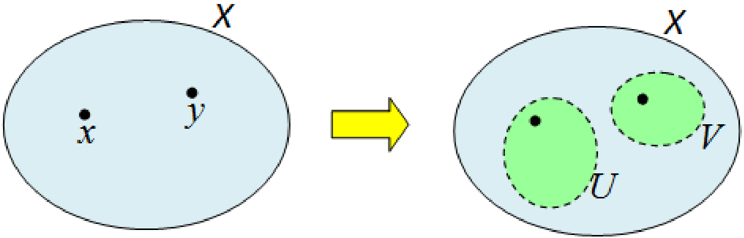

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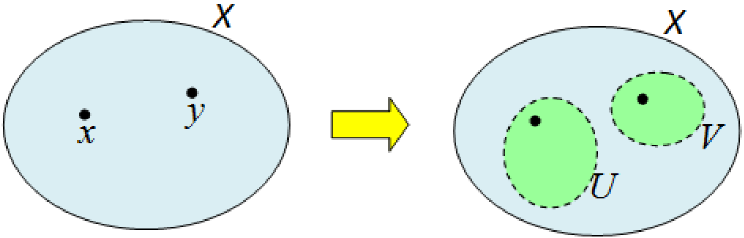

Metric spaces are trivially Hausdorff (take open balls around x, y with radius $\frac{d(x,y)}{2}$).

A non-Hausdorff space

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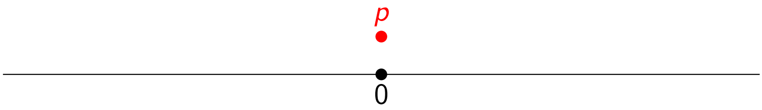
Consider the $\mathbb{R}$ line with two origins.



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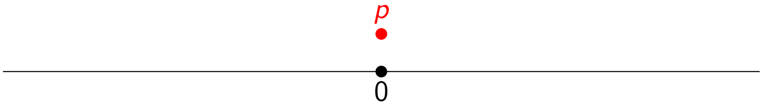

with τ topology generated by sets of the form (a, b) and $(a', 0) \cup \{p\} \cup (0, b')$.

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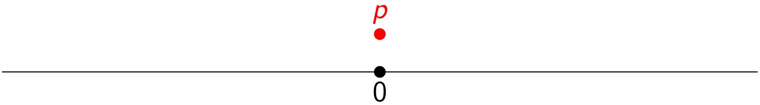

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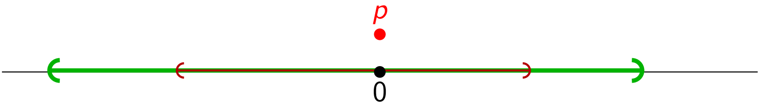

It's not Hausdorff, because every open set containing 0 (but not p) also intersects every open set containing p (but not 0).

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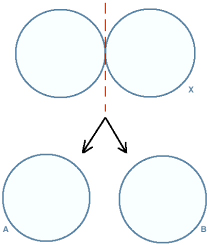

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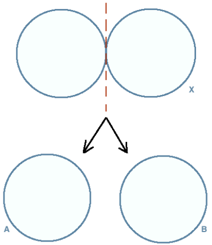

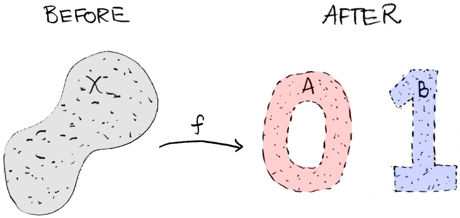

(X, τ) is disconnected if $X = A \cup B$, where $A, B \in \tau$ and $A \cap B = \emptyset$. X is connected if it's not disconnected.

(X, τ) is disconnected if there is some *continuous surjection* $f : (X, \tau) \to (\{0, 1\}, \mathcal{P}(\{0, 1\}))$. X is connected if it's not disconnected, i.e., every continuous $f : X \to \{0, 1\}$ is constant.

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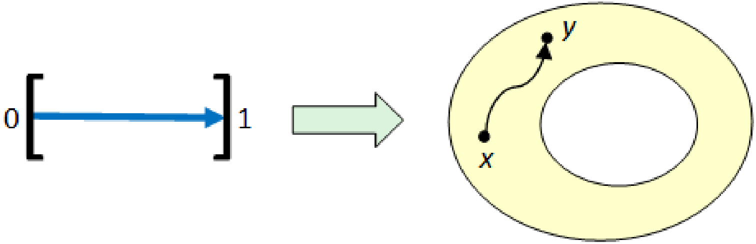

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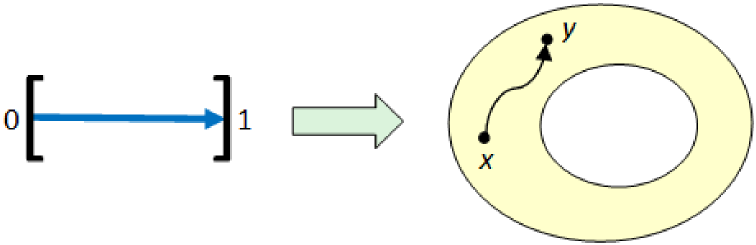

Note that it's a stronger condition than connectedness, since *disconnectedness* $\implies$ *non-path-connectedness* (there's a gap inbetween!).

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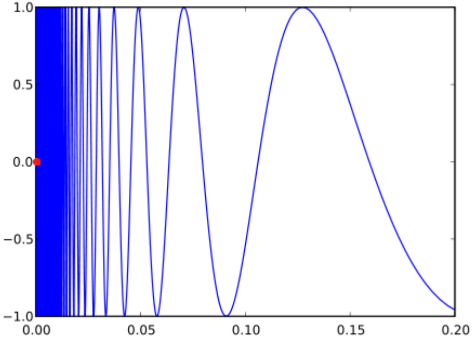
- Intervals $(a, b) \subset \mathbb{R}$ are path-connected (and, thus, connected);
- intervals of the form $(a, b) \cup (b, c) = (a, c) \setminus \{b\}$ are disconnected (and also fail to be path-connected).

Fish Out of Water: A Counterexample



The *topologist's sine curve* $\left(x, \sin(\frac{1}{x})\right) \cup \{0, 0\}$ is connected, but is *not* path-connected.

Simply Connected Spaces

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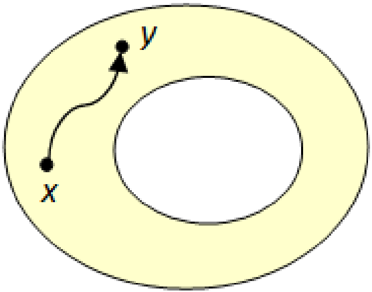
Spaces with "holes" fail to be simply connected.



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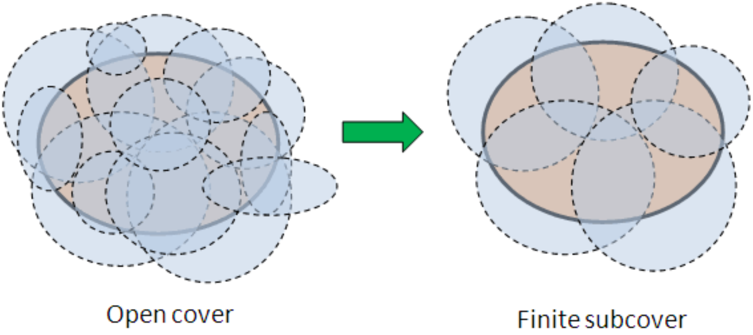

Compactness in $\mathbb{R}^n$

In $\mathbb{R}^n$, compact sets are closed and bounded (with respect to the Euclidean metric), of the form $\prod_{k=1}^{n} [a_k, b_k]$, and vice-versa (by Heine-Borel Theorem).

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A classic theorem from Calculus:
Weierstrass' Theorem: A continuous function on a closed and bounded set (i.e., compact) $f : K \subset \mathbb{R}^n \to \mathbb{R}$ is bounded, i.e., attains its maximum and minimum in K.

Manifolds and General Relativity

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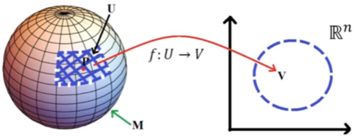

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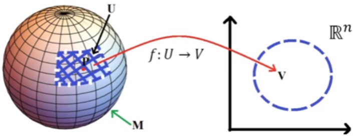

Spacetime in General Relativity is a 4-dimensional $\mathcal{C}^\infty$-differentiable manifold with a Lorentz metric.

Hilbert Spaces and Quantum Mechanics

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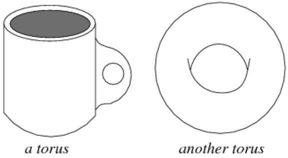
A Hilbert Space is a vector space $\mathcal{H}$ endowed with an inner product $<,>$, such that it's a complete metric space (with its induced metric) (i.e., Cauchy sequences converge in $\mathcal{H}$). Wavefunctions in Quantum Mechanics are unit vectors in $(\mathcal{H}, <, >)$, and observables (momentum, position, etc.) are Hermitian operators $O : \mathcal{H} \to \mathcal{H}$.

Obnoxious Math Jokes/Memes



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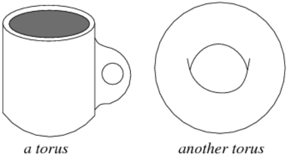

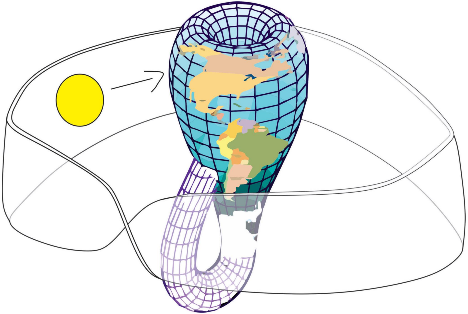

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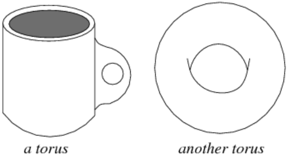

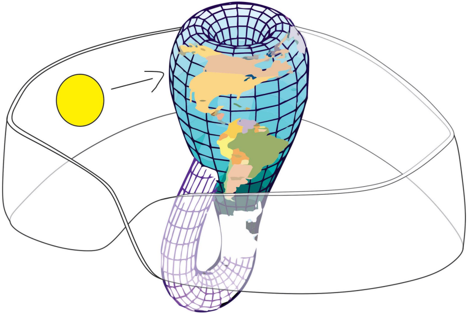

Guess that's enough for today.
Thank you for coming!