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Topology (and Metrics) for the Young at Heart

Nicholas Funari Voltani

March 21, 2019

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$\mathbb{R} \qquad \qquad \sqrt{2} \qquad e \ \pi$

Figure: Our beloved \mathbb{R} line.

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 $x \in \mathbb{R}$

 $|x| \in \mathbb{R}_+$

Absolute Value

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|x - y|: distance between $x, y \in \mathbb{R}$.

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•
$$|x-y| \neq 0 \iff x \neq y;$$

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- $|x-y| \neq 0 \iff x \neq y;$
- $|x z| \le |x y| + |y z|$ (Triangle Inequality);

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- $|x-y| \neq 0 \iff x \neq y;$
- $|x z| \le |x y| + |y z|$ (Triangle Inequality);
- $|x y| \ge 0$ (Positive distances);

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- $|x-y| \neq 0 \iff x \neq y;$
- $|x z| \le |x y| + |y z|$ (Triangle Inequality);
- $|x y| \ge 0$ (Positive distances);
- |x y| = |y x| (Commutativity)

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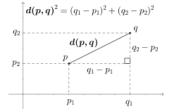
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Distances in \mathbb{R}^n

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Euclidean distance in \mathbb{R}^2

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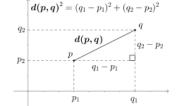
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Distances in \mathbb{R}^n

Same idea for \mathbb{R}^{n} ! (Just the size of the *n*-rectangle's diagonal)

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Euclidean distance in \mathbb{R}^2

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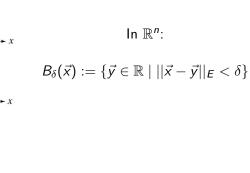
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In \mathbb{R} : a < x < b $x \in (a,b)$ - x Open interval $a \le x \le b$ $x \in [a,b]$ - x Closed interval

Open balls in \mathbb{R}^n



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Open sets in \mathbb{R}^n

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$A \subset \mathbb{R}^n$ is open if:

$$\forall x \in A, \exists \delta_x > 0 \mid B_{\delta_x}(x) \subset A.$$

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$A \subset \mathbb{R}^n$ is open if:

 $\forall x \in A, \exists \delta_x > 0 \mid B_{\delta_x}(x) \subset A.$ I.e., every point has an open ball around it entirely within A.

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It can be proven that

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It can be proven that

• Arbitrary unions of open sets is an open set;

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It can be proven that

- Arbitrary unions of open sets is an open set;
- Finite intersections of open sets is an open set;

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$A \subset \mathbb{R}^n$ is open if:

 $\forall x \in A, \exists \delta_x > 0 \mid B_{\delta_x}(x) \subset A.$

I.e., every point has an open ball around it entirely within A.

It can be proven that

- Arbitrary unions of open sets is an open set;
- Finite intersections of open sets is an open set;
 - Even a countable intersection may fail to be open: $\bigcap_{n\in\mathbb{N}}(-1,\frac{1}{n})=(-1,0]$

Open sets in \mathbb{R}^n

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Let $M \neq \emptyset$ be a set.

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Definition: Metric Spaces

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Let $M \neq \emptyset$ be a set. Metric: $d: M \times M \rightarrow \mathbb{R}$ such that, for every $x, y, z \in M$,

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Definition: Metric Spaces

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Let $M \neq \emptyset$ be a set. *Metric*: $d : M \times M \to \mathbb{R}$ such that, for every $x, y, z \in M$, • $d(x, y) = 0 \iff x = y$;

- $d(x,z) \le d(x,y) + d(y,z)$ (Triangle Inequality);
- *d*(*x*, *y*) ≥ 0 (Positivity);
- d(x, y) = d(y, x) (Commutativity)

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Definition: Metric Spaces

Let $M \neq \emptyset$ be a set. *Metric*: $d : M \times M \to \mathbb{R}$ such that, for every $x, y, z \in M$, • $d(x, y) = 0 \iff x = y$; • $d(x, z) \le d(x, y) + d(y, z)$ (Triangle Inequality);

•
$$d(x, y) = d(y, x)$$
 (Commutativity)

The pair (M, d) is called a *metric space* when d is a metric.

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Some Classic Metric Spaces

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$$(\mathbb{R}, |.|)$$
 and (\mathbb{R}^n, d_E) , $d_E(\vec{x}, \vec{y}) = \sqrt{\sum_{k=1}^n (x_i - y_i)^2}$

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 and (\mathbb{R}^n, d_E) , $d_E(\vec{x}, \vec{y}) = \sqrt{\sum_{k=1}^n (x_i - y_i)^2}$

•
$$M \neq \emptyset$$
;
 $d_t(x,y) = \begin{cases} 0, \text{if } x = y;\\ 1, \text{if } x \neq y \end{cases}$ (Trivial metric)

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 and (\mathbb{R}^n, d_E) , $d_E(\vec{x}, \vec{y}) = \sqrt{\sum\limits_{k=1}^n (x_i - y_i)^2}$

•
$$M \neq \emptyset$$
;
 $d_t(x,y) = \begin{cases} 0, \text{if } x = y; \\ 1, \text{if } x \neq y \end{cases}$ (Trivial metric)

•
$$(\mathbb{R}^n, d_1)$$
, where $d_1(\vec{x}, \vec{y}) := \sum_{i=1}^n |x_i - y_i|$

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$$(\mathbb{R}, |.|)$$
 and $(\mathbb{R}^n, d_E), \ d_E(\vec{x}, \vec{y}) = \sqrt{\sum_{k=1}^n (x_i - y_i)^2}$

•
$$M \neq \emptyset$$
;
 $d_t(x, y) = \begin{cases} 0, \text{if } x = y; \\ 1, \text{if } x \neq y \end{cases}$ (Trivial metric)

•
$$(\mathbb{R}^n, d_1)$$
, where $d_1(\vec{x}, \vec{y}) := \sum_{i=1}^n |x_i - y_i|$

• $(\mathcal{C}([a,b]),d_{\infty})$, where $d_{\infty}(f,g):=\sup_{x\in [a,b]}|f(x)-g(x)|$

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Pseudometrics: A case

Consider this snake-y boi, M.

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Pseudometrics: A case

Let $x, y \in M$.

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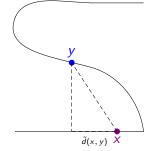
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Pseudometrics: A case

Consider $\tilde{d}(x, y)$ to be the horizontal "distance" between x and y.

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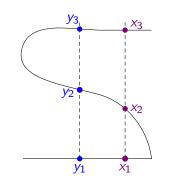
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Pseudometrics: A case

Distinct points above each other have "distance" 0!

 $\therefore \tilde{d}$ isn't a metric!

But it satisfies the other properties...

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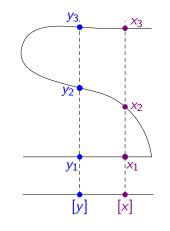
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Creating Metrics from Pseudometrics

Let's identify points:

$$a \sim b \iff \tilde{d}(a, b) = 0$$

$$[x] = \{z \in M \mid \tilde{d}(x, z) = 0\}$$
$$= \{z \in M \mid z \sim x\}$$

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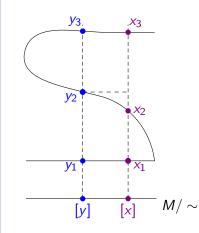
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Creating Metrics from Pseudometrics

Define $d([x], [y]) := \tilde{d}(x, y)$. It is well-defined, since

$$egin{aligned} & ilde{d}(x_1,y_1) \leq ilde{d}(x_2,y_2) \ & ilde{d}(x_2,y_2) \leq ilde{d}(x_1,y_1) \ & \therefore ilde{d}(x_1,y_1) = ilde{d}(x_2,y_2) \end{aligned}$$

 $\therefore (M/\sim, d)$ is a metric space.

What is it good for?

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We usually talk about convergent sequences with a metric.

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We usually talk about convergent sequences with a metric. $(x_n)_{n \in \mathbb{N}}$ is convergent to $x \in M \iff$ $\forall \epsilon > 0, \exists N_{\epsilon} \in \mathbb{N} \mid \forall n \geq N_{\epsilon}, d(x_n, x) < \epsilon.$

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We can talk about continuity of functions.

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We can talk about continuity of functions. $f: (M, d_M) \rightarrow (N, d_N)$ is continuous in $x \in M$ if $\forall \epsilon > 0, \exists \delta > 0 \mid 0 < d_M(x, y) < \delta \implies d_N(f(x), f(y)) < \epsilon.$

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What is it good for?

We can talk about continuity of functions. $f: (M, d_M) \rightarrow (N, d_N)$ is continuous in $x \in M$ if $\forall \epsilon > 0, \exists \delta > 0 \mid 0 < d_M(x, y) < \delta \implies d_N(f(x), f(y)) < \epsilon.$

There are naturally open balls of the form $B_{\delta}(x) = \{y \in M \mid d(x, y) < \delta\}.$ Definitions of open and closed sets is similar to that of \mathbb{R}^n .

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Let $X \neq \emptyset$. A set $\tau \subset \mathcal{P}(X)$ is called a *topology* if

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Let $X \neq \emptyset$. A set $\tau \subset \mathcal{P}(X)$ is called a *topology* if

• $\emptyset, X \in \tau;$

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Let $X \neq \emptyset$. A set $\tau \subset \mathcal{P}(X)$ is called a *topology* if

- $\emptyset, X \in \tau;$
- $\{A_{\lambda}\}_{\lambda\in\Lambda}\subset\tau\implies\bigcup_{\lambda\in\Lambda}A_{\lambda}\in\tau;$

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Definition: Topological Spaces

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Let $X \neq \emptyset$. A set $\tau \subset \mathcal{P}(X)$ is called a *topology* if

- $\emptyset, X \in \tau;$
- $\{A_{\lambda}\}_{\lambda\in\Lambda}\subset\tau\implies\bigcup_{\lambda\in\Lambda}A_{\lambda}\in\tau;$

•
$$\{A_k\}_{k=1}^n \subset \tau \implies \bigcap_{k=1}^n A_k \in \tau$$

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Definition: Topological Spaces

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•
$$\{A_k\}_{k=1}^n \subset \tau \implies \bigcap_{k=1}^n A_k \in \tau$$

The pair (X, τ) is called a *topological space* if τ is a topology for X.

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Open and Closed Sets

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Given (X, τ) topological space, $A \in \tau$ are called the *open sets* in X with respect to τ .

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Open and Closed Sets

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Given (X, τ) topological space, $A \in \tau$ are called the *open sets* in X with respect to τ .

Every $F \subset X \mid X \setminus F \in \tau$ are its *closed sets* (also w.r.t. τ). Let's call $\mathcal{F}(\tau)$ the collection of these closed sets.

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Given (X, τ) topological space, $A \in \tau$ are called the *open sets* in X with respect to τ .

Open and Closed Sets

Every $F \subset X \mid X \setminus F \in \tau$ are its *closed sets* (also w.r.t. τ). Let's call $\mathcal{F}(\tau)$ the collection of these closed sets.

Note that "open-ness" / "closed-ness" is always with respect to some topology (also obviously w.r.t the space X).

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Some Classic Topological Spaces

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• $(\mathbb{R}^n, \tau_{std})$ (surprise, surprise);

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Some Classic Topological Spaces

- $(\mathbb{R}^n, \tau_{std})$ (surprise, surprise);
- Sierpinski space $\mathbb{S} = \{0, 1\}, \tau_{\mathbb{S}} = \{\emptyset, \{1\}, \{0, 1\}\}$ is a topological space!

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Some Classic Topological Spaces

- $(\mathbb{R}^n, \tau_{std})$ (surprise, surprise);
- Sierpinski space $\mathbb{S} = \{0, 1\}, \tau_{\mathbb{S}} = \{\emptyset, \{1\}, \{0, 1\}\}$ is a topological space!
- Metric spaces (M, d), with their induced topology by the metric τ_d can be seen as topological spaces.

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Interior and Closure of Sets

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Given (X, τ) topological space, and $Z \subset X$, then

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Given
$$(X, au)$$
 topological space, and $Z\subset X$, then

$$\mathring{Z} := \bigcup_{\substack{A \in \tau \\ A \subset Z}} A$$

is called Z's interior

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$$\overline{Z} := \bigcap_{\substack{F \in \mathcal{F}(\tau) \\ Z \subset F}} F$$

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is called Z's closure

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Interior and Closure of Sets

Given
$$(X, au)$$
 topological space, and $Z\subset X$, then

$$\mathring{Z} := \bigcup_{\substack{A \in \tau \\ A \subset Z}} A$$

is called Z's interior (inflating Z with inner open sets).

$$\overline{Z} := \bigcap_{\substack{F \in \mathcal{F}(\tau) \\ Z \subset F}} F$$

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is called Z's closure

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is called Z's closure (enclosing Z with larger closed sets).

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Continuity in Topological Spaces

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A map $f: (X, \tau_X) \to (Y, \tau_Y)$ is said to be continuous if, for every $V \in \tau_Y$, $f^{-1}(V) \in \tau_X$. (In \mathbb{R}^n , it's equivalent to the $\epsilon - \delta$ definition.)

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Continuity in Topological Spaces

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A bijection $X \stackrel{\varphi}{\longleftrightarrow} Y$ is said to be a *homeomorphism* if it's continuous both ways.

We say $X \sim Y$ if there's a homeomorphism between them; they're said to be *topologically equivalent*.

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A bijection $X \stackrel{\varphi}{\longleftrightarrow} Y$ is said to be a *homeomorphism* if it's continuous both ways.

We say $X \sim Y$ if there's a homeomorphism between them; they're said to be *topologically equivalent*.

Homeomorphisms are important objects in Topology, since they preserve many topological properties.

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The Hausdorff property

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Given
$$(X, \tau)$$
 topological space, X is Hausdorff if
 $\forall x, y \in X, \exists U_x, U_y \in \tau \mid U_x \cap U_y = \emptyset, x \in U_x, y \in U_y$

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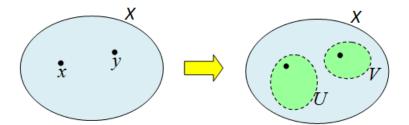
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The Hausdorff property

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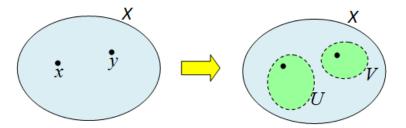
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Metric spaces are trivially Hausdorff (take open balls around x, y with radius $\frac{d(x,y)}{2}$).

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Topology (and Metrics) for the Young at Heart	A non-Hausdorff space
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A non-Hausdorff space

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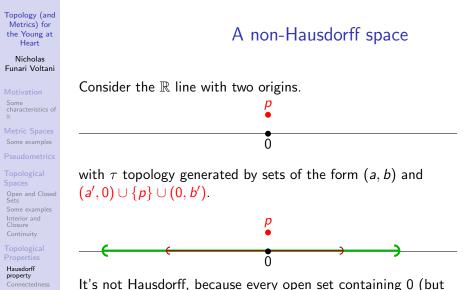
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Consider the $\ensuremath{\mathbb{R}}$ line with two origins.

with τ topology generated by sets of the form (a, b) and $(a', 0) \cup \{p\} \cup (0, b')$.

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not p) also intersects every open set containing p (but not 0).

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There are two equivalent definitions for disconnected (and connected) spaces:

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There are two equivalent definitions for disconnected (and connected) spaces:

 (X, τ) is disconnected if $X = A \cup B$, where $A, B \in \tau$ and $A \cap B = \emptyset$. X is connected if it's not disconnected.

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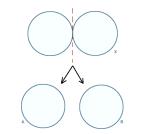
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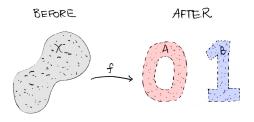
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Connected Spaces

There are two equivalent definitions for disconnected (and connected) spaces:



 (X, τ) is disconnected if $X = A \cup B$, where $A, B \in \tau$ and $A \cap B = \emptyset$. X is connected if it's not disconnected.



 (X, τ) is disconnected if there is some continuous surjection $f: (X, \tau) \rightarrow (\{0, 1\}, \mathcal{P}(\{0, 1\})).$ X is connected if it's not disconnected, i.e., every continuous $f: X \rightarrow \{0, 1\}$ is constant.

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Path-Connectedness

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 (X, τ) is *path-connected* if for every $x, y \in X$, there is a continuous path $\gamma : [0, 1] \to X, \gamma(0) = x, \gamma(1) = y$.

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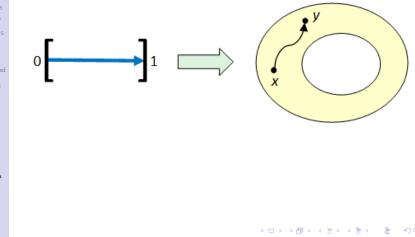
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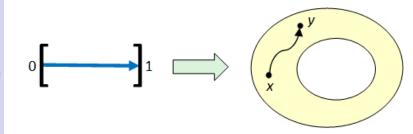
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Path-Connectedness

 (X, τ) is *path-connected* if for every $x, y \in X$, there is a continuous path $\gamma : [0, 1] \to X, \gamma(0) = x, \gamma(1) = y$.



Note that it's a stronger condition than connectedness, since $disconnectedness \implies non-path-connectedness$ (there's a gap inbetween!).

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Examples of [Path-]Connected Spaces

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Intervals (a, b) ⊂ ℝ are path-connected (and, thus, connected);

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Examples of [Path-]Connected Spaces

- Intervals (a, b) ⊂ ℝ are path-connected (and, thus, connected);
- intervals of the form (a, b) ∪ (b, c) = (a, c) \ {b} are disconnected (and also fail to be path-connected).

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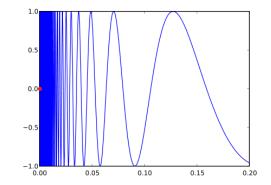
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Fish Out of Water: A Counterexample



The topologist's sine curve $(x, sin(\frac{1}{x})) \cup \{0, 0\}$ is connected, but is *not* path-connected.

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Simply Connected Spaces

A more usual concept in Calculus is that of simply connected spaces:

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A more usual concept in Calculus is that of simply connected spaces:

For every $x, y \in X$, every continuous path connecting them can be deformed into any other continuous path connecting them.

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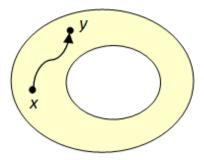
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Simply Connected Spaces

A more usual concept in Calculus is that of simply connected spaces:

For every $x, y \in X$, every continuous path connecting them can be deformed into any other continuous path connecting them. Spaces with "holes" fail to be simply connected.



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An open cover of $K \subset X$ is a collection of open sets $\{A_{\lambda}\}_{\lambda \in \Lambda}$ such that $K \subset \bigcup_{\lambda \in \Lambda} A_{\lambda}$.

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Topology (and Metrics) for the Young at Heart

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An open cover of $K \subset X$ is a collection of open sets $\{A_{\lambda}\}_{\lambda \in \Lambda}$ such that $K \subset \bigcup_{\lambda \in \Lambda} A_{\lambda}$.

A set $K \subset X$ is compact if, for every open cover $\{A_{\lambda}\}_{\lambda \in \Lambda} \subset \tau$, there is a finite open cover $\{A_k\}_{k=1}^n \subset \{A_{\lambda}\}_{\lambda \in \Lambda}$.

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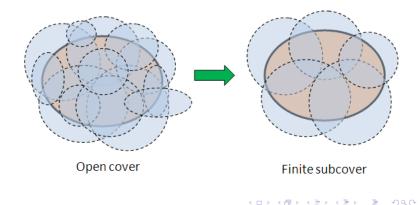
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Compactness in \mathbb{R}^n

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Compactness in \mathbb{R}^n

A classic theorem from Calculus:

Weierstrass' Theorem: A continuous function on a closed and bounded set (i.e., compact) $f : K \subset \mathbb{R}^n \to \mathbb{R}$ is bounded, i.e., attains its maximum and minimum in K.

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A locally Euclidean space (dimension n) is a topological space (X, τ) such that, for every $p \in X$, $\exists V_p \in \tau$ such that V_p is homeomorphic to an open disk in \mathbb{R}^n .

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A topological manifold (dimension *n*) is a triple (X, τ, A) ,

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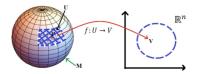
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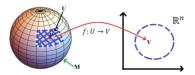
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Spacetime in General Relativity is a 4-dimensional C^{∞} -differentiable manifold with a Lorentz metric.



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Hilbert Spaces and Quantum Mechanics

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Given a vector space ${\mathcal H}$ with an inner product <,>, one can induce a norm by defining

 $||v|| := \sqrt{\langle v, v \rangle}$

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$$|\mathbf{v}|| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Given a norm, one can induce a metric:

$$d(u,v) := ||u-v||$$

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A Hilbert Space is a vector space \mathcal{H} endowed with an inner product \langle , \rangle , such that it's a complete metric space (with its induced metric) (i.e., Cauchy sequences converge in \mathcal{H}).

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Given a norm, one can induce a metric: d(u, v) := ||u - v||

A Hilbert Space is a vector space \mathcal{H} endowed with an inner product \langle , \rangle , such that it's a complete metric space (with its induced metric) (i.e., Cauchy sequences converge in \mathcal{H}). Wavefunctions in Quantum Mechanics are unit vectors in $(\mathcal{H}, \langle , \rangle)$, and observables (momentum, position, etc.) are Hermitian operators $O : \mathcal{H} \to \mathcal{H}$.

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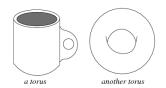
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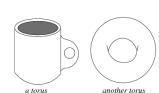
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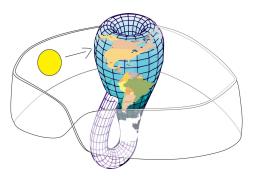
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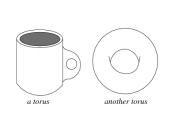
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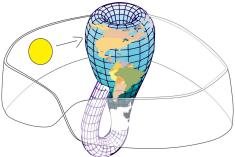
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Guess that's enough for today. Thank you for coming!