Mathematical Proofs

Applications on Physics and Discourse in General

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Outline

Logics and Truth Tables

The Language

The Language: Negation

p	$\neg p$
T	F
F	T

The Language: And

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The Language: Or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The Language: Implication

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Contradiction

If p implies a falsehood, p is, itself, false.

Contradiction: A Meme

More Language: Equivalence

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

More Language: Contrapositive

p	q	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Contraposition

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

A Simple Problem

If $x^2 - 6x + 5$ is even, then x is odd.

Proof.

By contraposition, it suffices to prove that if x isn't odd, then $x^2 - 6x + 5$ isn't even. In other words:
If x is even, then $x^2 - 6x + 5$ is odd.

$$x^2 - 6x + 5$$

□

The Irracionality of $\sqrt{2}$

$\sqrt{2}$ is irrational

Proof.

Let us assume, for a contradiction, that $\sqrt{2}$ is rational, that is: There exists p and q integer numbers such that $\sqrt{2} = \frac{p}{q}$.

$$2 = \frac{p^2}{q^2}$$

$$2q^2 = p^2$$

□

How Many Primes Are There?

There exists infinitely many prime numbers.

Proof.

Let us assume, for a contradiction, that there exists a finite number of primes and denote them by $p_1, \ldots, p_n$. Now, consider the number

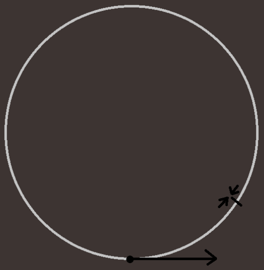
$$p \equiv p_1 \ldots p_n + 1$$

Noting that no prime divides p, we conclude that p must be prime itself. □

Equilibrium Points on a Rim

Consider a rim and a particle whose movement is constrained to it. The particle is subject a force field $\vec{F}(\vec{r})$ dependent only on the position of the particle along the rim, along with the contact forces with the rim itself. All equilibrium points are either stable or unstable. There is an even number of equilibrium points.

Proof.



What the Heck is Induction?

Consider $p(n)$ a proposition whose truth depends on a natural number n. If:

1. $p(0)$ is true.
2. $p(k) \Rightarrow p(k+1)$ for all k natural.

Then, $p(n)$ is true for all natural numbers.

A Very Simple Example

If $f : \mathbb{R} \to \mathbb{R}$ is such that $f(a+b) = f(a) + f(b)$, then, for every natural number n

$$f(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} f(x_i)$$

Proof.

1. For $n = 1$, we have to check whether $f(x_1) = f(x_1)$. This is left as an exercise.
2. Assume it is true that

$$f(\sum_{i=1}^{k} x_i) = \sum_{i=1}^{k} f(x_i)$$

□

A Very Simple Example

If $f : \mathbb{R} \to \mathbb{R}$ is such that $f(a+b) = f(a) + f(b)$, then, for every natural number n

$$f(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} f(x_i)$$

Proof.

We have to prove that

$$f(\sum_{i=1}^{k+1} x_i) = \sum_{i=1}^{k+1} f(x_i)$$

$$f(\sum_{i=1}^{k+1} x_i) = f(x_{k+1} + \sum_{i=1}^{k} x_i) = f(x_{k+1}) + f(\sum_{i=1}^{k} x_i)$$

□

A Very Simple Example

If $f : \mathbb{R} \to \mathbb{R}$ is such that $f(a+b) = f(a) + f(b)$, then, for every natural number n

$$f(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} f(x_i)$$

Proof.

$$= f(x_{k+1}) + \sum_{i=1}^{k} f(x_i) = \sum_{i=1}^{k+1} f(x_i)$$

□

Another Very Simple Example

$$S_n \equiv 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Proof.

1. For $n = 1$, we have to check if $1 = 1$. This is also left as an exercise.
2. Assume it's true for $n = k$, that is, assume $S_k = \frac{k(k+1)}{2}$. Now, consider $n = k + 1$. Note that

$$S_{k+1} = 1 + 2 + \cdots + k + (k+1) = S_k + (k+1)$$

By hypothesis,

$$S_{k+1} = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

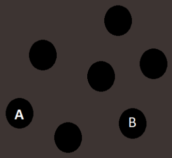
□

A Very Illegal Example

I will now prove that for every natural number n, every ball in every set of n balls has the same color as the other balls in the set. Which is very illegal.

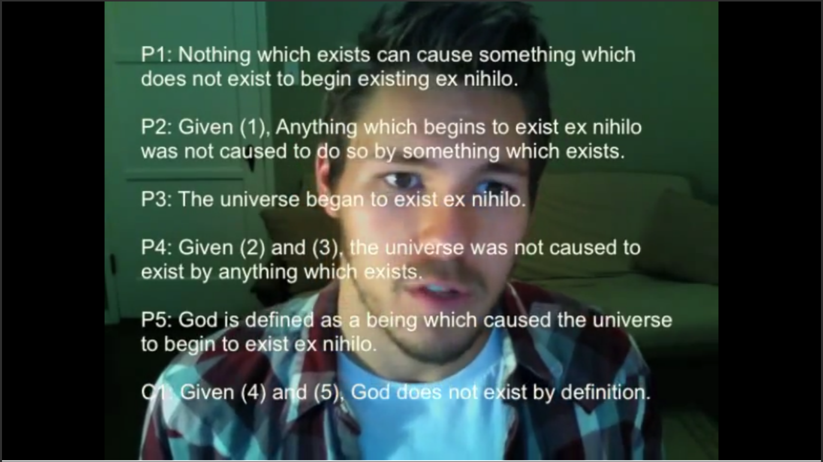
Proof.

1. It is trivially true for a set that only contains one ball.
2. The inductive step goes as follows



□

An Example in Debate



Acknowledgments

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The End