From Ebony to Ivory

Fourier Transforms and their applications to PDEs

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Summary

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- Maxwell's Electrodynamics
- Heisenberg's Uncertainty Principle

A Silly Idea

Ordinary Differential Equations

$$\frac{\mathrm{d}}{\mathrm{d}x}y(x) + \frac{1}{CR}y(x) = 0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) + \gamma\frac{\mathrm{d}}{\mathrm{d}x}y(x) + \omega_0^2 y(x) = f(x)$$

Ordinary Differential Equations

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Livin' La Vida Loca

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) + \gamma\frac{\mathrm{d}}{\mathrm{d}x}y(x) + \omega_0^2 y(x) = f(x)$$

$$\Downarrow$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \gamma\frac{\mathrm{d}}{\mathrm{d}x} + \omega_0^2\right] y(x) = f(x)$$

$$\Downarrow$$

$$y(x) = \frac{f(x)}{\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \gamma\frac{\mathrm{d}}{\mathrm{d}x} + \omega_0^2}$$

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Livin' La Vida Loca

Pandora's Box

$$f(x) \longrightarrow \boxed{\mathcal{F}} \longrightarrow \hat{f}(\xi)$$

Pandora's Box

$$(f + \alpha g)(x) \longrightarrow \boxed{\mathcal{F}} \longrightarrow \hat{f}(\xi) + \alpha \hat{g}(\xi)$$

Pandora's Box

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \longrightarrow \boxed{\mathcal{F}} \longrightarrow i\xi \hat{f}(\xi)$$

Pandora's Box

$$\hat{f}(\xi) \longrightarrow \boxed{\mathcal{F}^{-1}} \longrightarrow f(x)$$

Pandora's Box

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \gamma\frac{\mathrm{d}}{\mathrm{d}x} + \omega_0^2\right] y(x) = f(x)$$

$$\downarrow$$

$$\boxed{\mathcal{F}}$$

$$\downarrow$$

$$\left[-\xi^2 + i\gamma\xi + \omega_0^2\right] \hat{y}(\xi) = \hat{f}(\xi)$$

Box Proposal

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} \, \mathrm{d}x$$

$$\mathcal{F}^{-1}[\hat{f}](x) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{ix\xi} \, \mathrm{d}\xi$$

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Quality Control

$$\widehat{(f+\alpha g)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(f(x) + \alpha g(x)\right) e^{-ix\xi}\, \mathrm{d}x$$

$$\Downarrow$$

$$\widehat{(f+\alpha g)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi}\, \mathrm{d}x + \frac{\alpha}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-ix\xi}\, \mathrm{d}x$$

$$\Downarrow$$

$$\widehat{(f+\alpha g)}(\xi) = \hat{f}(\xi) + \alpha \hat{g}(\xi)$$

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$$(\widehat{f+\alpha g})(\xi) = \hat{f}(\xi) + \alpha \hat{g}(\xi)$$

Quality Control

$$\widehat{f'}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f'(x) e^{-ix\xi} \, \mathrm{d}x$$

$$\Downarrow$$

$$\widehat{f'}(\xi) = \left. \frac{f(x) e^{-ix\xi}}{\sqrt{2\pi}} \right|_{-\infty}^{+\infty} + i\xi \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} \, \mathrm{d}x$$

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$$\widehat{f'}(\xi) = i\xi \widehat{f}(\xi)$$

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$$\Downarrow$$

$$\widehat{f'}(\xi) = i\xi \widehat{f}(\xi)$$

Quality Control

The inverse does work

for appropriate functions

and, sometimes, the Fourier Transform of a function is not in the same set as the original function, but let's forget about this since we do not know a decent theory of integration

Playing Around With Our New Toy

Fourier Transforming

$$f(t) = \cos(\omega_0 t)e^{-\pi t^2}$$

$$\widehat{f}(\omega) = \frac{e^{-\frac{(\omega-\omega_0)^2}{4\pi}} + e^{-\frac{(\omega+\omega_0)^2}{4\pi}}}{2\sqrt{2\pi}}$$

$$\omega = 2\pi\nu$$

Fourier Transforming

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$$\omega = 2\pi\nu$$

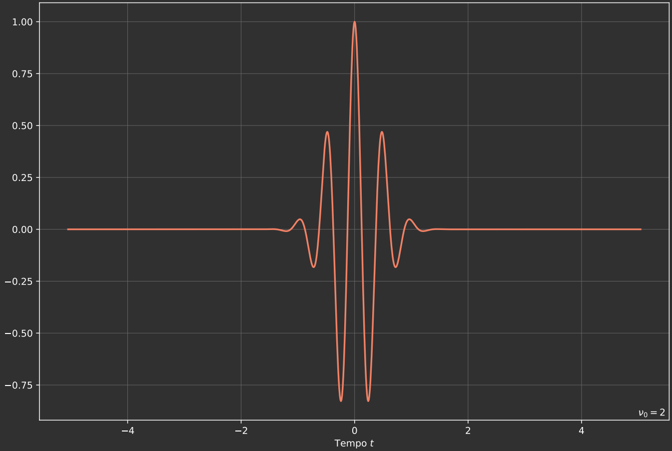
Fourier Transforming

$$f(t) = \cos(2\pi\nu_0 t)e^{-\pi t^2}$$

$$\widehat{f}(\nu) = \frac{e^{-\pi(\nu-\nu_0)^2} + e^{-\pi(\nu+\nu_0)^2}}{2\sqrt{2\pi}}$$

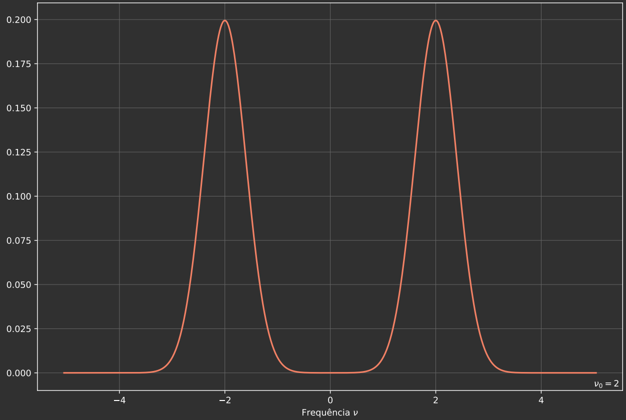
Fourier Transforming

$$f(t) = \cos(2\pi\nu_0 t)e^{-\pi t^2}$$



Fourier Transforming

$$\widehat{f}(\nu) = \frac{e^{-\pi(\nu-\nu_0)^2} + e^{-\pi(\nu+\nu_0)^2}}{2\sqrt{2\pi}}$$



A Harder Example

$$f(t) = e^{i\omega_0 t} = \cos(\omega_0 t) + i\sin(\omega_0 t)$$

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega_0 t} e^{-i\omega t} \, \mathrm{d}t$$

A Harder Example

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The Mathematical Moonwalk

$$f(t) = e^{i\omega_0 t}$$

$$e^{i\omega_0 t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{f}(\omega) e^{i\omega t} \, \mathrm{d}\omega$$

$$\widehat{f}(\omega) = \sqrt{2\pi}\delta\left(\omega - \omega_0\right)$$

The Mathematical Moonwalk

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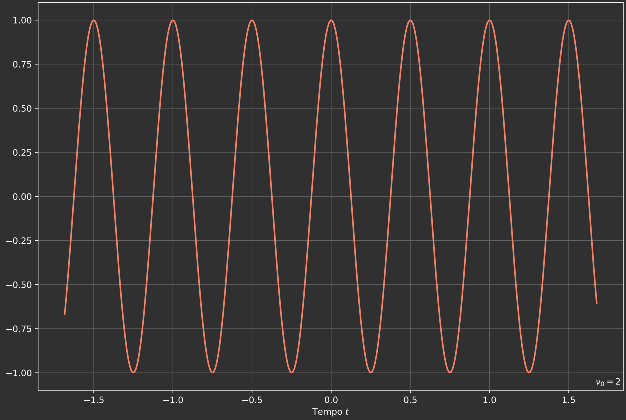
$$\widehat{f}(\omega) = \sqrt{2\pi} \delta \left(\omega - \omega_0\right)$$

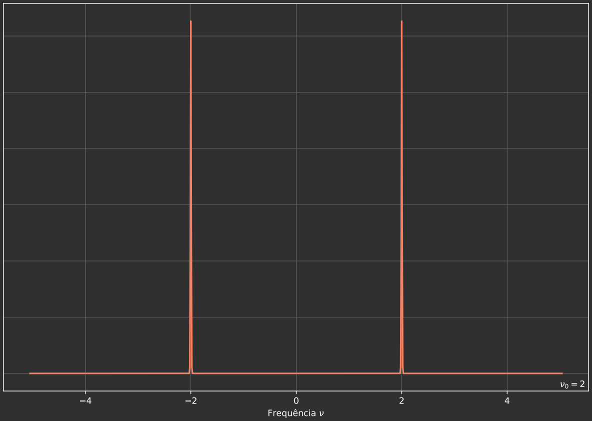
Cosines

$$f(t) = \cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$$



Cosines

$$\widehat{f}(\omega) = \sqrt{\frac{\pi}{2}}\left(\delta\left(\omega - \omega_0\right) + \delta\left(\omega + \omega_0\right)\right)$$



Fourier's Physics Playground
Maxwell's Electrodynamics

In the beggining, God said:

$$\begin{cases} \boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \\ \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \\ \boldsymbol{\nabla} \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

and there was light!

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and there was light!

Too hard, let's try something different

$$\begin{cases} \mathbf{E} = -\boldsymbol{\nabla} V - \dfrac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \end{cases}$$

Wave Equations

$$\begin{cases} \nabla^2 V - \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2} = -\dfrac{\rho}{\epsilon_0} \\ \nabla^2 \mathbf{A} - \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \end{cases}$$

All Wave Equations In One

$$\nabla^2\psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}(\mathbf{r},t) = -g(\mathbf{r},t)$$

Fourier's Opinion

$$\widehat{g}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\mathbf{r}, t) e^{-i\omega t} \, \mathrm{d}t$$

$$g(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{g}(\mathbf{r}, \omega) e^{i\omega t} \, \mathrm{d}\omega$$

Fourier's Opinion

$$\widehat{\psi}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(\mathbf{r}, t) e^{-i\omega t} \, \mathrm{d}t$$

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Fourier's Opinion

$$\nabla^2\psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}(\mathbf{r},t) = -g(\mathbf{r},t)$$

$$\nabla^2\widehat{\psi}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\widehat{\psi}(\mathbf{r},\omega) = -\widehat{g}(\mathbf{r},\omega)$$

Green Function

$$L\phi(\mathbf{r}) = -s(\mathbf{r})$$

$$LG(\mathbf{r}-\mathbf{r}') = -\delta\left(\mathbf{r}-\mathbf{r}'\right)$$

$$\phi(\mathbf{r}) = \int G(\mathbf{r}-\mathbf{r}')s(\mathbf{r}')\,\mathrm{d}\tau'$$

$$L\phi(\mathbf{r}) = \int LG(\mathbf{r}-\mathbf{r}')s(\mathbf{r}')\,\mathrm{d}\tau' = -\int \delta\left(\mathbf{r}-\mathbf{r}'\right)s(\mathbf{r}')\,\mathrm{d}\tau' = -s(\mathbf{r})$$

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One At a Time

$$\nabla^2 \widehat{\psi}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\widehat{\psi}(\mathbf{r},\omega) = -\widehat{g}(\mathbf{r},\omega)$$

$$\nabla^2 G(\mathbf{r}-\mathbf{r}') + \frac{\omega^2}{c^2} G(\mathbf{r}-\mathbf{r}') = -\delta\left(\mathbf{r}-\mathbf{r}'\right)$$

One At a Time

$$\nabla^2 \widehat{\psi}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \widehat{\psi}(\mathbf{r}, \omega) = -\widehat{g}(\mathbf{r}, \omega)$$

$$\nabla^2 G(\mathbf{r} - \mathbf{r}') + \frac{\omega^2}{c^2} G(\mathbf{r} - \mathbf{r}') = -\delta\left(\mathbf{r} - \mathbf{r}'\right)$$

Solution for $\mathbf{r} - \mathbf{r}' \neq \mathbf{0}$

$$\frac{1}{r}\frac{\mathrm{d}^2(rG)}{\mathrm{d}r^2} + k^2 G = 0$$

$$G(r) = \frac{A}{r}e^{\pm ikr}$$

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Recovering 0 Psychological Trauma

$$\nabla^2 G(\mathbf{r}-\mathbf{r}') + \frac{\omega^2}{c^2} G(\mathbf{r}-\mathbf{r}') = -\delta\left(\mathbf{r}-\mathbf{r}'\right)$$

$$A\int \nabla^2 \frac{1}{r}\,\mathrm{d}\tau' + 4\pi A \frac{\omega^2}{c^2}\int \frac{r^2}{r}\,\mathrm{d}r = -\int \delta\left(\mathbf{r}-\mathbf{r}'\right)\mathrm{d}\tau'$$

$$-4\pi A = -1$$

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$$-4\pi A = -1$$

Back To Our Problem

$$\widehat{\psi}(\mathbf{r},\omega) = \int G(\mathscr{r})\,\widehat{g}(\mathbf{r}',\omega)\,\mathrm{d}\tau'$$

$$G(\mathscr{r}) = \frac{1}{4\pi\mathscr{r}} e^{\pm ik\mathscr{r}}$$

$$\widehat{\psi}(\mathbf{r},\omega) = \frac{1}{4\pi} \int \frac{\widehat{g}(\mathbf{r}',\omega)e^{\pm ik\mathscr{r}}}{\mathscr{r}}\,\mathrm{d}\tau'$$

Back To Our Problem

$$\widehat{\psi}(\mathbf{r},\omega) = \int G(\mathscr{r})\widehat{g}(\mathbf{r}',\omega)\,\mathrm{d}\tau'$$

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Actually Solving Our Problem

$$\psi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{\psi}(\mathbf{r}, \omega) e^{i\omega t} \, \mathrm{d}\omega$$

$$\psi(\mathbf{r}, t) = \frac{1}{4\pi\sqrt{2\pi}} \iint \frac{\widehat{g}(\mathbf{r}', \omega) e^{i\omega t \pm i\omega \frac{\mathscr{r}}{c}}}{\mathscr{r}} \, \mathrm{d}\omega \, \mathrm{d}\tau'$$

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Actually Solving Our Problem

$$\psi(\mathbf{r},t) = \frac{1}{4\pi\sqrt{2\pi}} \iint \frac{\widehat{g}(\mathbf{r}',\omega)e^{i\omega\left(t\pm\frac{\mathscr{r}}{c}\right)}}{\mathscr{r}}\,\mathrm{d}\omega\,\mathrm{d}\tau'$$

$$\psi(\mathbf{r},t) = \frac{1}{4\pi} \int \frac{g(\mathbf{r}',t\pm\frac{\mathscr{r}}{c})}{\mathscr{r}}\,\mathrm{d}\tau'$$

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$$\psi(\mathbf{r}, t) = \frac{1}{4\pi} \int \frac{g\left(\mathbf{r}', t - \frac{\mathscr{r}}{c}\right)}{\mathscr{r}} \, \mathrm{d}\tau'$$

Back at Maxwell's

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t - \frac{\mathscr{r}}{c})}{\mathscr{r}} \, \mathrm{d}\tau'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t - \frac{\mathscr{r}}{c})}{\mathscr{r}} \, \mathrm{d}\tau'$$

One Last Step

$$\begin{cases} \mathbf{E} = -\boldsymbol{\nabla} V - \dfrac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \end{cases}$$

Jefimenko Equations

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\lfloor\rho\rfloor + \frac{\hat{\boldsymbol{\mathscr{r}}}}{c\,\mathscr{r}}\left\lfloor\frac{\partial\rho}{\partial t}\right\rfloor - \frac{1}{c^2\,\mathscr{r}}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor \mathrm{d}\tau'$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int\left(\frac{1}{\mathscr{r}^2}\lfloor\mathbf{J}\rfloor + \frac{1}{c\,\mathscr{r}}\left\lfloor\frac{\partial\mathbf{J}}{\partial t}\right\rfloor\right)\times\hat{\boldsymbol{\mathscr{r}}}\,\mathrm{d}\tau'$$

Fourier's Physics Playground
Heisenberg's Uncertainty Principle

Position and Momentum

$$\psi(x) = \langle x|\psi\rangle = \int \langle x|k\rangle \langle k|\psi\rangle \, \mathrm{d}k = \frac{1}{\sqrt{2\pi}} \int e^{ikx} \psi(k) \, \mathrm{d}k$$

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Position and Momentum (but weirder)

$$\begin{cases} X\left|\psi\right\rangle = x\psi(x) \\ K\left|\psi\right\rangle = -i\dfrac{\partial\psi}{\partial x}(x) \end{cases}$$

$$\begin{cases} X\left|\psi\right\rangle = -i\dfrac{\partial\psi}{\partial k}(k) \\ K\left|\psi\right\rangle = k\psi(k) \end{cases}$$

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$$\begin{cases} X\left|\psi\right\rangle = -i\dfrac{\partial\psi}{\partial k}(k) \\ K\left|\psi\right\rangle = k\psi(k) \end{cases}$$

Fourier Diplomacy

$$|x\rangle \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} |k\rangle$$

Fourier Uncertainty

1. $\psi(x)$: what is x?
2. $\psi(k)$: what is k?

Fourier Uncertainty

1. $\psi(x)$: what is x?
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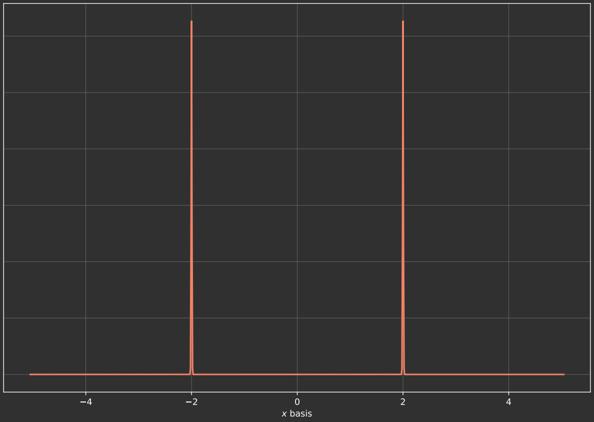
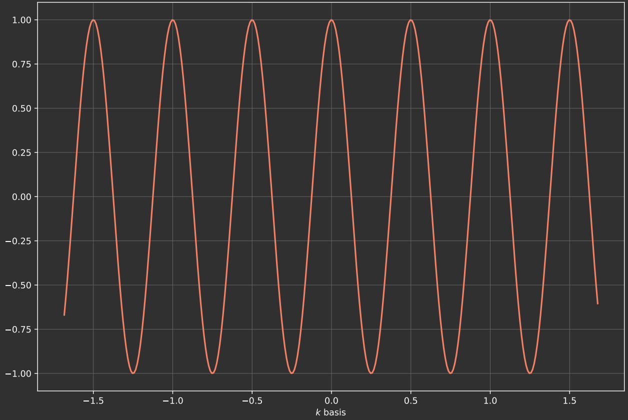
Definite Position

$$\psi(x) = \sqrt{\frac{\pi}{2}}\left(\delta\left(x - x_0\right) + \delta\left(x + x_0\right)\right)$$



Undefinite Momentum

$$\psi(k) = \cos(x_0 k)$$



Uncertainty Relation

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

The uncertainty relation is a consequence of the general fact that anything narrow in one space is wide in the transform space and vice versa. So if you are a 45 kg weakling and are taunted by a 270 kg bully, just ask him to step into momentum space!
Ramamurti Shankar

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